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Patching and Thickening Problems

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INTRODUCTION

In recent years, much progress has been made on the structure of fundamental groups of algebraic curves by means of patching techniques in formal and rigid geometry. A number of these results have concerned curves over algebraically closed fields of characteristic p, e.g., the proofs of the Abhyankar Conjecture ([Ra], [Ha5]) and of the geometric case of the Shafarevich Conjecture ([Po1], [Ha6]), and the realization of Galois groups over projective curves ([Sa1], [St1]).

While the rigid approach to patching is often regarded as more intuitive than the formal approach, its foundations are less well-established. But constructions involving the formal approach have tended to be technically more cumbersome. The purpose of the current paper is to build on previous formal patching results to create a framework in which such constructions are facilitated. In the process we prove a result asserting that singular curves over a field k can be thickened to curves over $k[[t]]$ with prescribed behavior in a formal neighborhood of the singular locus, and

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similarly for covers of curves. Afterward, we obtain applications to fundamental groups of curves over large fields.

The structure of the paper is as follows. Section 1 concerns patching problems for projective curves X^* over a power series ring $R = k[[t_1, \ldots, t_n]]$. It is shown (Theorem 1) that giving a coherent projective module over X^* is equivalent to giving such modules compatibly on a formal neighborhood of each singular point of the closed fiber X, and on the formal thickening along the complement of the singular locus S of X. Section 2 applies this to thickening problems, in both the absolute and relative senses. Namely, it shows (Theorem 3) that such an X^* can be constructed from its closed fiber X and from complete local thickenings near S, such that X^* is a trivial deformation away from S. Moreover (Theorem 2), given a morphism $X \to Z$ and a thickening Z^* of Z, such that the local thickenings near S are compatible with that of Z, there is a unique thickening X^* of X that is compatible with the given data. In Section 3 these results are applied to the problem of thickening and deforming covers. Theorem 4 there combines the results of Sections 1 and 2 to show that covers of reducible k-curves can be thickened to covers of curves over $k[[t]]$, and Theorem 5 then shows how this can be used over large fields to construct Galois covers with desired Galois groups and inertia groups, in many cases, over a generic curve of a given genus. This result is applied in Section 4 to obtain information about the structure of fundamental groups of curves. In the affine case, we obtain (Theorem 6) a simple proof of a key result needed in Raynaud's proof of the Abhyankar Conjecture for the affine line, without the use of rigid machinery such as Runge pairs used in [Ra]. We state this result for the case of large fields, rather than just for algebraically closed fields. We then prove a result on finite quotients of π_1 of curves with prescribed ramification, over large fields (Theorem 7). Corollaries 1–4 of Theorem 7 provide examples of this.

Finally, we fix some terminology that will be used in the paper. If A is an algebra over a ring R, then A will be called *generically separable* if its total ring of fractions is separable over that of R, and no non-zero divisor of R becomes a zero divisor of A. A morphism ϕ: $X \to Z$ of schemes is *generically separable* if over each affine open subset $U = \operatorname{Spec} R$ of Z, the corresponding R-algebra is generically separable. If ϕ is both finite and generically separable, then we call it a *cover*. If G is a finite group, then by a *G-Galois cover* we will mean a cover ϕ: $X \to Z$ together with a homomorphism ι: $G \to \operatorname{Aut}_Z(X)$, with respect to which G acts simply transitively on each generic geometric fiber of ϕ. If Z is irreducible, then a cover $X \to Z$ is *Galois* if X is also irreducible and $X \to Z$ is $\operatorname{Aut}_Z(X)$-Galois. If (Z, ζ) is a pointed irreducible scheme, then the pointed Galois étale covers of Z, with base points over ζ, form an inverse system whose automorphism group $\pi_1(Z, \zeta)$ is the *algebraic fundamental group* of (Z, ζ).

Up to isomorphism, this group is independent of the choice of the base point ζ, and the reference to ζ is usually suppressed. We also let $\pi_A(Z)$ denote the set of (isomorphism classes of) continuous finite quotients of $\pi_1(Z)$. Thus a finite group G is in $\pi_A(Z)$ if and only if G is the Galois group of a Galois étale cover of Z. If we are working over a given field k, and if g is a nonnegative integer, then $\pi_A(g)$ will denote the set of finite groups G for which there is a dense open subset M_G in the moduli space $\mathscr{M}_g$ of curves of genus g, such that $G \in \pi_A(U)$ for every U corresponding to a k-point of M_G.

1. PATCHING PROBLEMS

Over the complex numbers, covers of curves can be constructed by cutting and pasting in the metric topology. Over more general fields, this method does not apply, and the Zariski topology is too weak to patch together open subsets of distinct covers. But by working over a complete field, such as $k((t))$, one can recapture some of the machinery of the complex situation. There, one can either work with the t-adic metric and rigid analytic spaces, or with schemes over $k[[t]]$ and formal geometry. Here we take the latter approach.

The formal approach began with Zariski's Theorem on Formal Functions, and came to fruition with Grothendieck's Existence Theorem [Gr1, Corollaire 5.1.6]. As a result, it is possible to construct Galois covers of a given k-curve by giving the cover locally in the formal topology and then patching. (Compare [Ha3], which realized every finite group as a Galois group of a branched cover of the line over any non-archimedean local field.) For certain constructions, though, it is necessary to give some of the data even more locally, viz. at the complete local ring of a point, rather than along open subsets of the closed fiber. For this reason, a formal patching theorem allowing such patchings was proved in [Ha4, Theorem 1] and used to realize certain groups as Galois groups with certain types of specified ramification. Still more involved constructions (e.g., those of [Ha5] and [St1]) require the use of $k[[t]]$-curves with reducible fibers. While these situations can be reduced to that of [Ha4, Theorem 1], the reduction leads to much lengthier and more involved proofs. In the present section we derive from [Ha4, Theorem 1] a more general result that can be used more easily in patching constructions. Later in the paper, we use this result in obtaining results about thickenings of curves and about fundamental groups in characteristic p.

Following the notation of [Ha2], for any scheme T, let $\mathscr{M}(T)$ denote the category of coherent $\mathscr{O}_T$-modules. Similarly, let $\mathscr{F}(T)$ [resp. $\mathscr{P}(T)$] denote the subcategory of $\mathscr{M}(T)$ consisting of free [resp. projective] modules.

Also, let $\mathscr{A}(T), \mathscr{AF}(T), \mathscr{AP}(T)$ denote the categories of coherent $\mathscr{O}_T$-algebras which, as $\mathscr{O}_T$-modules, lie in $\mathscr{M}(T), \mathscr{F}(T), \mathscr{P}(T)$, respectively. Also let $\mathscr{S}(T), \mathscr{SF}(T), \mathscr{SP}(T)$ denote the corresponding categories of generically separable algebras, and for any finite group G let $G(T), G\mathscr{F}(T), G\mathscr{P}(T)$ denote the corresponding categories of G-Galois $\mathscr{O}_T$-algebras.

Let k be a field, let m be a nonnegative integer, and let $R = k[[t_1, \dots, t_m]]$, with maximal ideal $I = (t_1, \dots, t_m)$. Let X^* be a connected projective normal R-curve, let X be its closed fiber, and let S be a nonempty finite subset of X that contains all of the singular points of X. (Thus $X - S$ is a regular affine curve.) For any affine open subset $U \subset X - S$, we may consider the ring of formal functions on X^* along U, viz. $\hat{\mathscr{O}}_{X^*,U} = \lim_{\leftarrow} \tilde{A}/I^n\tilde{A}$, where $\operatorname{Spec} \tilde{A} = \tilde{U}$ is any affine open subset of X^* with closed fiber U. In this situation, define a *module patching problem* $\overline{M}$ for (X^*, S) to consist of the following:

(i) a finite $\hat{\mathscr{O}}_{X^*,U}$-module M_U for every irreducible component U of $X - S$;

(ii) a finite $\hat{\mathscr{O}}_{X^*,\xi}$-module M_ξ for every $\xi \in S$;

(iii) an $\hat{\mathscr{O}}_{X^*,\xi,\wp}$-module isomorphism $\mu_{U,\xi,\wp}\colon M_U \otimes_{\hat{\mathscr{O}}_{X^*,U}} \mathscr{O}_{X^*,\xi,\wp} \xrightarrow{\sim} M_\xi \otimes_{\hat{\mathscr{O}}_{X^*,\xi}} \hat{\mathscr{O}}_{X^*,\xi,\wp}$ for each choice of $U, \xi, \wp$ where ξ lies in S, $\wp$ is a minimal prime of $\hat{\mathscr{O}}_{X^*,\xi}$ containing I, U is the closure of the point $\wp$ in $X - S$ (i.e., the irreducible component of $X - S$ containing $\wp$), and $\hat{\mathscr{O}}_{X^*,\xi,\wp}$ is the completion of the localization of $\hat{\mathscr{O}}_{X^*,\xi}$ at $\wp$.

A *morphism* of module patching problems for (X^*, S) consists of morphisms between the corresponding M_U's and M_ξ's which are compatible with the $\mu_{U,\xi,\wp}$'s. Thus the module patching problems for (X^*, S) form a category, which we denote by $\mathscr{M}(X^*, S)$. Similarly, we may define the notions of *projective module patching problem*, *algebra patching problem*, etc., and the corresponding categories $\mathscr{P}(X^*, S), \mathscr{A}(X^*, S)$, etc.

Note that there is a natural "base change" functor $\beta_S\colon \mathscr{M}(X^*) \to \mathscr{M}(X^*, S)$, and similarly for $\mathscr{P}(X^*, S)$, $\mathscr{A}(X^*, S)$, etc. Namely, β_S assigns to each object M in $\mathscr{M}(X^*)$ the induced modules $M_U = M \otimes_{\mathscr{O}_{X^*}} \hat{\mathscr{O}}_{X^*,U}$ for each U and $M_\xi = M \otimes_{\mathscr{O}_{X^*}} \hat{\mathscr{O}}_{X^*,\xi}$ for each ξ, along with the induced isomorphisms $\mu_{U,\xi,\wp}$. A *solution* to a module patching problem $\overline{M}$ in $\mathscr{M}(X^*, S)$ is an object in $\mathscr{M}(X^*)$ that maps to $\overline{M}$ under the base change functor (and similarly for the other types of patching problems). The main theorem of this section asserts, in particular, that every projective module patching problem has a unique solution, up to isomorphism (see below).

If X^*, X, S are as above, and if S' is a finite subset of X containing S, then every patching problem for (X^*, S) induces a patching problem for (X^*, S'). More precisely, the induced patching problem is the image of the given one under a certain functor $\gamma_{S,S'}\colon \mathscr{M}(X^*, S) \to \mathscr{M}(X^*, S')$ [resp.

$\mathscr{P}, \mathscr{A}$, etc.], which we now define. Specifically, suppose we are given a module patching problem $\overline{M} = (\{M_U\}, \{M_\xi\}, \{\mu_{U,\xi,\wp}\})$. Then each irreducible component U' of $X - S'$ lies in a unique irreducible component U of $X - S$, and we define $M'_{U'} = M_U \otimes_{\hat{\mathscr{O}}_{X^*,U}} \hat{\mathscr{O}}_{X^*,U'}$. Also, if $\xi \in S$, let $M'_\xi = M_\xi$; while if $\xi \in S' - S$, then ξ is a smooth point on a unique irreducible component U of $X - S$, and let $M'_\xi = M_U \otimes_{\hat{\mathscr{O}}_{X^*,U}} \hat{\mathscr{O}}_{X^*,\xi}$. Finally, we need to define $\mu'_{U',\xi,\wp}$ for each $\xi \in S'$ and each minimal prime $\wp$ of $\hat{\mathscr{O}}_{X^*,\xi}$ containing (t), where U' is the irreducible component of $X - S'$ containing $\wp$. If $\xi \in S$, then we may consider the open set U as in (iii) above, and the natural map $j: M_U \to M'_{U'}$. Then j induces an isomorphism:

$$j_*: M_U \otimes_{\hat{\mathscr{O}}_{X^*,U}} \hat{\mathscr{O}}_{X^*,\xi,\wp} \xrightarrow{\sim} M'_{U'} \otimes_{\hat{\mathscr{O}}_{X^*,U'}} \hat{\mathscr{O}}_{X^*,\xi,\wp}, \tag{1}$$

and so we may take $\mu'_{U',\xi,\wp} = \mu_{U,\xi,\wp} \circ j_*^{-1}$. On the other hand, if $\xi \in S' - S$, then by the above definitions of M'_U and M'_ξ we have natural identifications,

$$i_{U'}: M'_{U'} \otimes_{\hat{\mathscr{O}}_{X^*,U'}} \hat{\mathscr{O}}_{X^*,\xi,\wp} \xrightarrow{\sim} M_U \otimes_{\hat{\mathscr{O}}_{X^*,U'}} \hat{\mathscr{O}}_{X^*,\xi,\wp}$$

and

$$i_\xi: M'_\xi \otimes_{\hat{\mathscr{O}}_{X^*,\xi}} \hat{\mathscr{O}}_{X^*,\xi,\wp} \xrightarrow{\sim} M_U \otimes_{\hat{\mathscr{O}}_{X^*,U'}} \hat{\mathscr{O}}_{X^*,\xi,\wp},$$

and we may take $\mu'_{U',\xi,\wp} = i_\xi^{-1} \circ i_{U'}$. So in each of these cases we have defined the modules $M'_{U'}, M'_\xi$ and the isomorphisms $\mu'_{U',\xi,\wp}$, and we then define the desired $\gamma_{S,S'}: \mathscr{M}(X^*, S) \to \mathscr{M}(X^*, S')$ by

$$\gamma_{S,S'}\big(\{M_U\}, \{M_\xi\}, \{\mu_{U,\xi,\wp}\}\big) = \big(\{M'_{U'}\}, \{M'_\xi\}, \{\mu'_{U',\xi,\wp}\}\big). \tag{2}$$

The constructions in the cases of $\mathscr{P}, \mathscr{A}$, etc. are similar.

It is easy to check that $\gamma_{S,S'}$ really is a functor, and that

$$\gamma_{S,S'} \circ \beta_S = \beta_{S'}: \mathscr{M}(X^*) \to \mathscr{M}(X^*, S).$$

Moreover, we have the following:

LEMMA. *With X^*, S, S' as above, the functor $\gamma_{S,S'}$ is faithful.*

Proof. We first show that $\gamma_{S,S'}$ is injective on isomorphism classes. So let

$$\overline{M} = \big(\{M_U\}, \{M_\xi\}, \{\mu_{U,\xi,\wp}\}\big)$$

be a patching problem for (X^*, S), and let

$$\overline{M}' = \left(\{M'_{U'}\}, \{M'_\xi\}, \{\mu'_{U',\xi,\wp}\}\right) = \gamma_{S,S'}(\overline{M})$$

as in (2) above. We claim that $\overline{M}$ is determined up to isomorphism by $\overline{M}'$.

Namely, first note that for all $\xi \in S$, $M_\xi = M'_\xi$. Next, for any irreducible component U of X_S, let $T_U = (S' - S) \cap U$. Also let

$$\hat{\mathscr{O}}_{X^*, T_U} = \prod_{\xi \in T_U} \hat{\mathscr{O}}_{X^*, \xi}$$

and

$$\hat{\mathscr{K}}_{X^*, T_U} = \prod_{\xi \in T_U} \hat{\mathscr{O}}_{X^*, \xi, \wp}.$$

Then we have an exact sequence,

$$0 \to \hat{\mathscr{O}}_{X^*, U} \xrightarrow{\Delta} \hat{\mathscr{O}}_{X^*, U'} \times \hat{\mathscr{O}}_{X^*, T_U} \xrightarrow{-} \hat{\mathscr{K}}_{X^*, T_U} \to 0,$$

where Δ is the diagonal inclusion and where the arrow labeled $-$ is given by subtraction as elements of $\hat{\mathscr{K}}_{X^*, T_U}$ (where we view $\hat{\mathscr{O}}_{X^*, U'}$ as contained in $\hat{\mathscr{K}}_{X^*, T_U}$ via a diagonal map Δ). Tensoring over $\hat{\mathscr{O}}_{X^*, U}$ with M_U and using that $\hat{\mathscr{K}}_{X^*, T_U}$ is flat over $\hat{\mathscr{O}}_{X^*, U}$, we obtain the exact sequence

$$0 \to M_U \xrightarrow{\Delta} M'_{U'} \times \prod_{\xi \in T_U} M'_\xi \xrightarrow{-} \prod_{\wp} M'_\wp \to 0,$$

where $\wp$ ranges over the minimal primes containing (t) in $\hat{\mathscr{O}}_{X^*, \xi}$ for all $\xi \in T_U$; where $M'_\wp = M'_\xi \otimes_{\hat{\mathscr{O}}_{X^*, \xi}} \hat{\mathscr{O}}_{X^*, \xi, \wp}$ for such pairs $(\wp, \xi)$; and where the subtraction in $M'_\wp$ takes place after applying $\mu'_{U', \xi, \wp}$ to the element of $M'_{U'}$. So M_U is determined, as a submodule of $M'_{U'} \times \prod_{\xi \in T_U} M'_\xi$, by the patching problem for (X^*, S'). Thus the map $j\colon M_U \to M'_{U'}$ is also determined, and hence so is the isomorphism j_* (as in (1) above) for each $\xi \in S$. Since $\mu'_{U', \xi, \wp} = \mu_{U, \xi, \wp} \circ j_*^{-1}$, we have that $\mu_{U, \xi, \wp}$ is determined as well.

It remains to show that $\gamma_{S,S'}$ is injective on morphisms. So suppose that

$$\phi = \left(\{\phi_U\}, \{\phi_\xi\}\right)\colon \left(\{M_U\}, \{M_\xi\}, \{\mu_{U,\xi,\wp}\}\right) \to \left(\{N_U\}, \{N_\xi\}, \{\nu_{U,\xi,\wp}\}\right)$$

is a morphism in $\mathscr{M}(X^*, S)$, inducing a morphism

$$\phi' = \left(\{\phi'_{U'}\}, \{\phi'_\xi\}\right)\colon \left(\{M'_{U'}\}, \{M'_\xi\}, \{\mu'_{U',\xi,\wp}\}\right) \to \left(\{N'_{U'}\}, \{N'_\xi\}, \{\nu'_{U,\xi,\wp}\}\right)$$

in $\mathscr{M}(X^*, S')$. By the previous paragraph (with N replacing M), we have that the morphism $\Delta\colon N_U \to N'_{U'} \times \prod_{\xi \in T_U} N'_\xi$ is injective. Hence the morphism $\phi_U\colon M_U \to N_U$ is determined by ϕ', as is $\phi_\xi\colon M_\xi \to N_\xi$ for $\xi \in S' - S$. Moreover, for $\xi \in S$, we have that $M_\xi = M'_\xi$, $N_\xi = N'_\xi$, and $\phi_\xi = \phi'_\xi$; so the morphism $\phi_\xi\colon M_\xi \to N_\xi$ is trivially determined by ϕ'. Thus ϕ' determines ϕ. ∎

We now have the following generalization of [Ha4, Theorem 1(3)], where as above $R = k[[t_1, \dots, t_m]]$, with maximal ideal $I = (\underline{t})$ (where we write $\underline{t}$ for $t_1, \dots, t_m$):

THEOREM 1 (Patching Theorem). *Let X^* be a connected projective normal R-curve, let X be its closed fiber, and let S be a nonempty finite subset of X that contains all of the singular points of X.*

(a) *Then the base change functor $\beta_S\colon \mathscr{P}(X^*) \to \mathscr{P}(X^*, S)$ is an equivalence of categories.*

(b) *The corresponding assertions hold for the categories $\mathscr{AP}$ of finite projective algebras and for the categories $G\mathscr{P}$ of projective G-Galois algebras for any finite group G.*

Proof. (a) Since X^* is projective, say $X^* \subset \mathbf{P}^n_R$, by taking a generic projection we obtain a finite morphism $f\colon X^* \to \mathbf{P}^1_R$. After composing f with a suitable finite morphism $g\colon \mathbf{P}^1_R \to \mathbf{P}^1_R$, we may assume that f maps the points of S to the point ∞ at infinity on the closed fiber of $\mathbf{P}^1_R$. Let $S' \subset X$ be the fiber over $\infty \in \mathbf{P}^1_k \subset \mathbf{P}^1_R$. Thus S' is a nonempty finite subset of X that contains S. Since $\gamma_{S,S'} \circ \beta_S = \beta_{S'}$, and since the functor $\gamma_{S,S'}$ is faithful, it suffices to prove the theorem with S' replacing S. So we are reduced to the case that $S = f^{-1}(\infty)$. Thus there are forgetful functors $f_*\colon \mathscr{P}(X^*) \to \mathscr{P}(\mathbf{P}^1_R)$ and $\bar{f}_*\colon \mathscr{P}(X^*, S) \to \mathscr{P}(\mathbf{P}^1_R, \{\infty\})$.

We first prove that β_S induces a bijection on isomorphism classes of objects. So given an object $\overline{M} = (\{M_U\}, \{M_\xi\}, \{\mu_{U,\xi,\wp}\})$ in $\mathscr{P}(X^*, S)$, we obtain an element $\bar{f}_*(\overline{M}) = \overline{N} = (N_{\mathbf{A}^1_k}, N_\infty, \nu)$ in $\mathscr{P}(\mathbf{P}^1_R, \{\infty\})$. Here $N_{\mathbf{A}^1_k} = \prod_U M_U$, viewed as a $k[x][[\underline{t}]]$-module; $N_\infty = \prod_\xi M_\xi$, viewed as a $k[[x^{-1}, \underline{t}]]$-module; and

$$\nu\colon N_{\mathbf{A}^1_k} \otimes_{k[x][[\underline{t}]]} k((x^{-1}))[[\underline{t}]] \xrightarrow{\sim} N_\infty \otimes_{k[[x^{-1},\underline{t}]]} k((x^{-1}))[[\underline{t}]]$$

is the $k((x^{-1}))[[\underline{t}]]$-isomorphism induced by $\{\mu_{U,\xi,\wp}\}$. Now the base change functors $\beta_\infty\colon \mathscr{P}(\mathbf{P}^1_R) \to \mathscr{P}(\mathbf{P}^1_R, \{\infty\})$ and $\beta_\infty^{\mathscr{A}}\colon \mathscr{AP}(\mathbf{P}^1_R) \to \mathscr{AP}(\mathbf{P}^1_R, \{\infty\})$ are equivalences of categories by [Ha4, Theorem 1(3)] and by the version for that result for $\mathscr{AP}$. So up to isomorphism there is a unique projective coherent $\mathscr{O}_{\mathbf{P}^1_R}$-module N in $\mathscr{P}(\mathbf{P}^1_R)$ such that $\beta_\infty(N) = \overline{N}$. Let $\mathscr{E}\!nd(N) \in \mathscr{AP}(\mathbf{P}^1_R)$ be the sheaf of endomorphisms of N. Similarly, we may consider $\mathscr{E}\!nd(\overline{N}) \in \mathscr{AP}(\mathbf{P}^1_R, \{\infty\})$, which satisfies $\beta_\infty^{\mathscr{A}}(\mathscr{E}\!nd(N)) = \mathscr{E}\!nd(\overline{N})$.

Let $\bar{\mathscr{O}}_{X^*} \in \mathscr{AP}(X^*, S)$ be the image of $\mathscr{O}_{X^*}$ under $\beta_S^{\mathscr{A}}$: $\mathscr{AP}(X^*) \to \mathscr{AP}(X^*, S)$. Then to give $\bar{M}$ in $\mathscr{P}(X^*, S)$ is equivalent to giving its image $\bar{N} = \bar{f}_*(\bar{M})$ in the category $\mathscr{P}(\mathbf{R}^1_R, \{\infty\})$, together with a homomorphism $\bar{\alpha}$: $\bar{\mathscr{O}}_{X^*} \to \mathscr{E}nd(\bar{N})$ in $\mathscr{AP}(\mathbf{P}^1_{k[[t]]}, \{\infty\})$ corresponding to the module structure over (X^*, S). Since $\beta_\infty^{\mathscr{A}}$ is an equivalence of categories, it follows that $\bar{\alpha}$ is induced by a unique homomorphism α: $\mathscr{O}_{X^*} \to \mathscr{E}nd(N)$ in $\mathscr{AP}(\mathbf{P}^1_{k[[t]]})$, where N is as above. Giving the pair (N, α) is equivalent to giving an $\mathscr{O}_{X^*}$-module M in $\mathscr{P}(X^*)$, and it is immediate that $\beta_S(M) \approx \bar{M}$. Moreover, M is unique up to isomorphism, since (as observed above) $\bar{N}$ determines N up to isomorphism, and since $\bar{\alpha}$ determines α. Thus β_S is indeed bijective on isomorphism classes of objects.

It remains to show that β_S induces a bijection on morphisms. So let M_1, M_2 be objects in $\mathscr{P}(X^*)$, and let $\bar{M}_i = \beta_S(M_i)$ in $\mathscr{P}(X^*, S)$ for $i = 1, 2$. Also, let $N_i = f_*(M_i)$ in $\mathscr{P}(\mathbf{P}^1_{k[[t]]})$ and let $\bar{N}_i = \bar{f}_*(\bar{M}_i)$ in $\mathscr{P}(\mathbf{P}^1_{k[[t]]}, \{\infty\})$. As before, let $\bar{\alpha}_i$: $\bar{\mathscr{O}}_{X^*} \to \mathscr{E}nd(\bar{N}_i)$ be the homomorphism in $\mathscr{AP}(\mathbf{P}^1_{k[[t]]}, \{\infty\})$ corresponding to the module structure of $\bar{M}_i$ over (X^*, S), and similarly for α_i: $\mathscr{O}_{X^*} \to \mathscr{E}nd(N_i)$ in $\mathscr{AP}(\mathbf{P}^1_{k[[t]]})$. If ψ: $N_1 \to N_2$ is a morphism in $\mathscr{P}(\mathbf{P}^1_{k[[t]]})$, then by composing with ψ on the left and right, respectively, we obtain induced morphisms ψ_*: $\mathscr{E}nd(N_1) \to \mathscr{H}om(N_1, N_2)$ and ψ^*: $\mathscr{E}nd(N_2) \to \mathscr{H}om(N_1, N_2)$ in $\mathscr{P}(\mathbf{P}^1_{k[[t]]})$. Similarly, if $\bar{\psi}$: $\bar{N}_1 \to \bar{N}_2$ is a morphism in $\mathscr{P}(\mathbf{P}^1_{k[[t]]}, \{\infty\})$, then by composing with $\bar{\psi}$ we obtain morphisms $\bar{\psi}_*$: $\mathscr{E}nd(N_1) \to \mathscr{H}om(\bar{N}_1, \bar{N}_2)$ and $\bar{\psi}^*$: $\mathscr{E}nd(\bar{N}_2) \to \mathscr{H}om(\bar{N}_1, \bar{N}_2)$ in $\mathscr{P}(\mathbf{P}^1_{k[[t]]}, \{\infty\})$. Observe that to give a morphism $\bar{\phi}$: $\bar{M}_1 \to \bar{M}_2$ in $\mathscr{P}(X^*, S)$ is equivalent to giving a morphism $\bar{\psi}$: $\bar{N}_1 \to \bar{N}_2$ in $\mathscr{P}(\mathbf{P}^1_{k[[t]]}, \{\infty\})$ such that $\bar{\psi}_* \circ \bar{\alpha}_1 = \bar{\psi}^* \circ \bar{\alpha}_2$. Similarly, giving a morphism ϕ: $M_1 \to M_2$ in $\mathscr{P}(X^*)$ is equivalent to giving a morphism ψ: $N_1 \to N_2$ in $\mathscr{P}(\mathbf{P}^1_{k[[t]]})$ such that $\psi_* \circ \alpha_1 = \psi^* \circ \alpha_2$.

Now let $\bar{\phi}$: $\bar{M}_1 \to \bar{M}_2$ be a morphism in $\mathscr{P}(X^*, S)$, and let $\bar{\psi}$: $\bar{N}_1 \to \bar{N}_2$ be the morphism in $\mathscr{P}(\mathbf{P}^1_{k[[t]]}, \{\infty\})$ induced by forgetting the (X^*, S)-structure of ϕ. By the previous paragraph, we have that $\bar{\psi}_* \circ \bar{\alpha}_1 = \bar{\psi}^* \circ \alpha_2$. Since β_∞: $\mathscr{P}(\mathbf{P}^1_{k[[t]]}) \to \mathscr{P}(\mathbf{P}^1_{k[[t]]}, \{\infty\})$ is an equivalence of categories, there is a unique morphism ψ: $N_1 \to N_2$ in $\mathscr{P}(\mathbf{P}^1_{k[[t]]})$ that induces $\bar{\psi}$, and this morphism satifies $\psi_* \circ \alpha_1 = \psi^* \circ \alpha_2$. Hence ψ is induced by a (unique) morphism ϕ: $M_1 \to M_2$ in $\mathscr{P}(X^*)$ via f_*. Here ϕ induces $\bar{\phi}$ via β_S, because ψ induces $\bar{\psi}$ and α_i induces $\bar{\alpha}_i$. Since any map in $\mathrm{Hom}(M_1, M_2)$ that induces $\bar{\phi} \in \mathrm{Hom}(\bar{M}_1, \bar{M}_2)$ must induce $\bar{\psi} \in \mathrm{Hom}(\bar{N}_1, \bar{N}_2)$ and hence $\psi \in \mathrm{Hom}(N_1, N_2)$ (using that β_∞ is an equivalence of categories), it follows that ϕ is the unique morphism in $\mathrm{Hom}(M_1, M_2)$ that induces $\bar{\phi}$. So indeed β_S is an equivalence of categories.

(b) This follows purely formally from part (a), as in the proofs of [Ha2, Proposition 2.8] and [Ha4, Theorem 1]. ∎

Remark. The requirement above that the modules be projective (or equivalently, flat) is due solely to the corresponding requirement of [Ha4, Theorem 1(3)]. If the latter requirement can be eliminated, then so can the former.

COROLLARY. *Under the hypotheses of the above theorem, with $m \le 1$, and with $U = X - S$, let $Y_U \to \operatorname{Spec} \hat{\mathscr{O}}_{X^*,U}$ and $Y_\xi \to \hat{\mathscr{O}}_{X^*,\xi}$ (for all $\xi \in S$) be normal covers [resp. G-Galois normal covers]. Suppose that for every $\xi \in S$, and for every minimal prime $\wp$ of $\hat{\mathscr{O}}_{X^*,\xi}$ containing I, an isomorphism is given between the covers that Y_U and Y_ξ induce over $k(\wp)[[\underline{t}]]$. Then there is a unique normal cover [resp. G-Galois normal cover] $Y \to X^*$ that induces the given covers, compatibly with the above identifications.*

Proof. This follows from Theorem 1 in the same way that [Ha4, Proposition 4(b)] followed from [Ha4, Theorem 1]. ∎

2. THICKENING PROBLEMS

In [Gr2, Corollaire 7.4], Grothendieck showed that if X is a smooth proper curve over a field k, and if R is a complete local ring with residue field k, then X is the closed fiber of a smooth proper R-curve. Thus each such X has a thickening over R. In this section, we consider the situation in which the given curve X is not necessarily smooth. Here, we show for $R = k[[t]]$ that if we are given thickenings near the singular points of X, then there is an R-curve with closed fiber X and inducing the given local thickenings. This is proved in the relative case in Theorem 2 (where the thickening is shown to be unique) and in the absolute case in Theorem 3. The proofs use Theorem 1 from Section 1. We begin by introducing terminology.

Let X be a projective k-curve that is connected and reduced (but not necessarily irreducible), and let S be a nonempty finite closed subset of X containing the singular locus. Let $X' = X - S$. By a *thickening* of (X, S) we will mean a projective normal $k[[t]]$-curve X^* together with an isomorphism $\phi\colon X \xrightarrow{\sim} X^*_{(t)}$ from X to the closed fiber of X^*, such that the restriction $\phi'\colon X' \xrightarrow{\sim} X'^*_{(t)}$ extends to a trivialization $(X' \times_k k[[t]])\hat{} \xrightarrow{\sim} X'^*$ away from S. Here $X'^* = \operatorname{Spec} \hat{\mathscr{O}}_{X^*,X'}$ (cf. Section 1).

By an *isomorphism* of thickenings $(X^*, \phi) \xrightarrow{\sim} (\tilde{X}^*, \tilde{\phi})$ of (X, S) we will mean an isomorphism $I\colon X^* \xrightarrow{\sim} \tilde{X}^*$ of $k[[t]]$-schemes such that $\tilde{\phi} = I_{(t)} \circ \phi$. Here $I_{(t)}\colon X^*_{(t)} \to \tilde{X}^*_{(t)}$ is the morphism induced by I via pullback.

Meanwhile, by a *thickening problem* Θ for (X, S) we will mean the assignment, for each closed point $\xi \in S$, of

(i) a Noetherian normal complete local domain R_ξ containing $k[[t]]$, such that t lies in the maximal ideal of R_ξ; and

(ii) a k-algebra isomorphism $F_\xi\colon R_\xi/(t) \xrightarrow{\sim} \hat{\mathscr{O}}_{X,\xi}$.

Here, F_ξ corresponds to an isomorphism $\phi_\xi\colon \operatorname{Spec} \hat{\mathscr{O}}_{X,\xi} \xrightarrow{\sim} \operatorname{Spec} R_\xi/(t)$ of k-schemes.

Let $\Theta = \{(R_\xi, F_\xi)\}$ and $\tilde{\Theta} = \{(\tilde{R}_\xi, \tilde{F}_\xi)\}$ be thickening problems for (X^*, S). By an *isomorphism* $\rho\colon \Theta \xrightarrow{\sim} \tilde{\Theta}$ we will mean a collection of $k[[t]]$-isomorphisms $\rho_\xi\colon R_\xi \xrightarrow{\sim} \tilde{R}_\xi$, for $\xi \in S$, such that $F_\xi = \tilde{F}_\xi \circ \bar{\rho}_\xi\colon R_\xi/(t) \xrightarrow{\sim} \hat{\mathscr{O}}_{X,\xi}$ for all $\xi \in S$, where $\bar{\rho}_\xi\colon R_\xi/(t) \xrightarrow{\sim} \tilde{R}_\xi/(t)$ is the map induced by ρ_ξ modulo (t).

With (X, S) as above, any thickening of (X, S) induces a thickening problem for (X, S). Namely, if (X^*, ϕ) is a thickening of (X, S), then let $\tilde{R}_\xi = \hat{\mathscr{O}}_{X^*,\xi}$ for each $\xi \in S$; let $\tilde{\phi}_\xi\colon \operatorname{Spec} \hat{\mathscr{O}}_{X,\xi} \xrightarrow{\sim} \operatorname{Spec} \tilde{R}_\xi/(t)$ be the pullback of ϕ to $\operatorname{Spec} \hat{\mathscr{O}}_{X,\xi}$; and let $\tilde{F}_\xi\colon \tilde{R}_\xi/(t) \xrightarrow{\sim} \hat{\mathscr{O}}_{X,\xi}$ be the corresponding k-algebra isomorphism. We then call $\Theta_{(X^*,\phi)} = \{(\tilde{R}_\xi, \tilde{F}_\xi)\}$ the *induced* thickening problem for (X, S). We also say that a thickening (X^*, ϕ) of (X, S) is a *solution* to a thickening problem Θ if Θ is isomorphic to the induced thickening problem $\Theta_{(X^*,\phi)}$.

Theorem 3 below guarantees a solution to such thickening problems. This is done by first proving an analogous result in the relative situation (Theorem 2, below), which relies on the patching results in Section 1 for projective algebras. The relative result is in a sense stronger, in that the solution in that situation is unique (up to isomorphism). Before stating that result, we consider the relative analogs of the terms that were defined above in the absolute situation.

So let Z^* be a connected projective normal $k[[t]]$-curve whose closed fiber Z is a connected, reduced, projective k-curve. Suppose that (Z^*, id) is a thickening of (Z, S_Z), for some $S_Z \subset Z$. Let X be another connected, reduced, projective k-curve; let $\psi\colon X \to Z$ be a finite generically separable morphism; and suppose that $S_X := \psi^{-1}(S_Z)$ contains the singular locus of X. Write $Z' = Z - S_Z$ and $X' = X - S_X$. By a *thickening* of (X, S_X) *relative to* $\psi\colon X \to Z$ and the inclusion $j\colon Z \hookrightarrow Z^*$ (or for short, a *relative thickening* of (X, S_X), if the morphisms ψ and j are understood) we will mean a thickening (X^*, ϕ) of (X, S_X) (in the above absolute sense), together with a finite morphism $\psi^*\colon X^* \to Z^*$, such that the closed fiber of ψ^* is $j \circ \psi \circ \phi^{-1}$, and such that $\psi^*\colon X^* \to Z^*$ is a trivial deformation of $\psi\colon X \to Z$ away from S_X and S_Z. This last condition means the following. Let X'^* and Z'^* be the completions of X^* and Z^* along $X' = X - S_X$ and $Z' = Z - S_Z$, and let $\psi'\colon X' \to Z'$ and $\psi'^*\colon X'^* \to Z'^*$ be the pullbacks of ψ and ψ^* over Z' and Z'^*. Then the content of the condition is that there are trivializations $(X' \times_k k[[t]])^\wedge \xrightarrow{\sim} X'^*$ and $(Z' \times_k k[[t]])^\wedge \xrightarrow{\sim} Z'^*$ that carry ψ'^* to the morphism $(X' \times_k k[[t]])^\wedge \to (Z' \times_k k[[t]])^\wedge$ induced by ψ'. By an *isomorphism* of relative thickenings $(X^*, \phi,$

$\psi^*) \tilde{\to} (\tilde{X}^*, \tilde{\phi}, \tilde{\psi}^*)$ of (X, S_X) as above, we will mean an isomorphism I: $(X^*, \phi) \tilde{\to} (\tilde{X}^*, \tilde{\phi})$ of absolute thickenings such that $\psi^* = \tilde{\psi}^* \circ I$.

Similarly, by a *thickening problem for* (X, S_X) *relative to* ψ: $X \to Z$ and j: $Z \hookrightarrow Z^*$ as above, we will mean a thickening problem $\{(R_\xi, F_\xi)\}$ for (X, S_X) together with a finite injective $k[[t]]$-algebra homomorphism ι_ξ: $\hat{\mathscr{O}}_{Z^*, \psi(\xi)} \hookrightarrow R_\xi$ for each $\xi \in S_X$, whose reduction $\bar{\iota}_\xi$: $\hat{\mathscr{O}}_{Z, \psi(\xi)} \hookrightarrow R_\xi/(t)$ modulo (t) has the property that $F_\xi \circ \bar{\iota}_\xi$: $\hat{\mathscr{O}}_{Z, \psi(\xi)} \hookrightarrow \hat{\mathscr{O}}_{X, \xi}$ is the map induced by completely localizing ψ at ξ. Suppose we are given relative thickening problems $\Theta = \{(R_\xi, F_\xi, \iota_\xi)\}$ and $\tilde{\Theta} = \{(\tilde{R}_\xi, \tilde{F}_\xi, \tilde{\iota}_\xi)\}$. Then by an *isomorphism* $\Theta \tilde{\to} \tilde{\Theta}$ of relative thickening problems for (X^*, S_X) we will mean an isomorphism $\rho = \{\rho_\xi\}$: $\{(R_\xi, F_\xi)\} \tilde{\to} \{(\tilde{R}_\xi, \tilde{F}_\xi)\}$ of the corresponding absolute thickening problems, such that $\tilde{\iota}_\xi = \rho_\xi \circ \iota_\xi$ for each $\xi \in S_X$.

As for absolute thickenings, each relative thickening induces a relative thickening problem. Namely, let (X^*, ϕ, ψ^*) be a thickening of (X, S_X) relative to ψ: $X \to Z$ and j: $Z \hookrightarrow Z^*$. Let $\{(R_\xi, F_\xi)\}$ be the (absolute) thickening problem for (X, S_X) induced by (X^*, ϕ). For each $\xi \in S_X$ let ι_ξ: $\hat{\mathscr{O}}_{Z^*, \zeta} \to R_\xi$ be the $k[[t]]$-algebra homomorphism induced by ψ^*, where $\zeta = \psi(\xi)$. Then ι_ξ is injective because ψ is generically separable, and the *induced relative thickening problem* $\Theta_{(X^*, \phi, \psi^*)}$ is given by $\{(R_\xi, F_\xi, \iota_\xi)\}$. As before, we say that a relative thickening (X^*, ϕ, ψ^*) of (X, S_X) is a *solution* to a relative thickening problem Θ if Θ is isomorphic to the induced relative thickening problem $\Theta_{(X^*, \phi, \psi^*)}$.

Relative thickening problems can be interpreted in terms of patching problems for Z^* (cf. Section 1); and in the next result we use Theorem 1 of Section 1 to find a unique solution to a given relative thickening problem. To do this we construct finite modules over rings of the form $\hat{\mathscr{O}}_{Z^*, U}$ and $\hat{\mathscr{O}}_{Z^*, \zeta}$ in such a way that they are compatible over each $\hat{\mathscr{O}}_{Z^*, \zeta, \wp}$ (cf. Section 1); here $\zeta \in S_Z$, and $\wp$ ranges over the minimal primes of $\hat{\mathscr{O}}_{Z^*, \zeta}$ containing the ideal (t). To ease the notation, we will denote $\hat{\mathscr{O}}_{Z^*, \zeta, \wp}$ simply by $\hat{\mathscr{O}}_{Z^*, \wp}$.

THEOREM 2. *Every relative thickening problem has a solution, which is unique up to isomorphism.*

Proof. Let ψ: $X \to Z$, j: $Z \hookrightarrow Z^*$, S_X, and S_Z be as above. Consider the map that assigns, to each thickening of (X, S_X) relative to ψ, j, its induced relative thickening problem. Then the theorem is equivalent to the assertion that this map is a bijection on isomorphism classes.

For surjectivity, let $\Theta = \{(R_\xi, F_\xi, \iota_\xi)\}$ be a thickening problem for (X, S_X) relative to ψ, j. For each $\zeta \in S_Z$, let R_ζ be the direct product of the rings R_ξ, where ξ ranges over $\psi^{-1}(\zeta)$. This is an $\hat{\mathscr{O}}_{Z^*, \zeta}$-algebra via the inclusions ι_ξ, and there is an $\hat{\mathscr{O}}_{Z,\zeta}$-algebra isomorphism $F_\zeta = \{F_\xi\}$: $R_\zeta/(t) \tilde{\to} \hat{\mathscr{O}}_{X,\zeta} := \prod_{\xi \in \psi^{-1}(\zeta)} \hat{\mathscr{O}}_{X,\xi}$. For each prime ideal $\wp$ of $\hat{\mathscr{O}}_{Z^*,\zeta}$ that is minimal among those containing (t), we may consider the complete localization $R_{\zeta,\wp}$ of the $\hat{\mathscr{O}}_{Z^*,\zeta}$-algebra R_ζ at $\wp$. The isomorphism F_ζ induces an isomorphism $F_{\zeta,\wp}$: $R_{\zeta,\wp}/(t) \tilde{\to} \hat{\mathscr{O}}_{X,\bar{\wp}} := \prod_{\bar{P} \in \psi^{-1}(\bar{\wp})} \hat{\mathscr{O}}_{X,\bar{P}}$ of $\hat{\mathscr{O}}_{Z,\bar{\wp}}$-algebras, where $\bar{\wp}$ is the reduction of $\wp$ modulo (t). As in Section 1, let $U = \operatorname{Spec} A$ be the closure in $Z - S_Z$ of the point corresponding to $\bar{\wp}$ (or equivalently, corresponding to $\wp$). Thus U is the irreducible component of $Z - S_Z$ "containing $\bar{\wp}$", and there is an inclusion $A \hookrightarrow \hat{\mathscr{O}}_{Z,\bar{\wp}}$. Similarly, there is an inclusion $B \hookrightarrow \hat{\mathscr{O}}_{X,\bar{\wp}}$, where $V = \operatorname{Spec} B = \psi^{-1}(U) \subset X$, and we have $\hat{\mathscr{O}}_{X,\bar{\wp}} = B \otimes_A \hat{\mathscr{O}}_{Z,\bar{\wp}}$. Since ψ is generically separable, the inclusion $\hat{\mathscr{O}}_{Z,\bar{\wp}} \hookrightarrow \hat{\mathscr{O}}_{X,\bar{\wp}}$ is étale. So by [Gr2, I, Corollaire 6.2], the $\hat{\mathscr{O}}_{Z,\bar{\wp}}$-algebra isomorphism $F_{\zeta,\wp}$: $R_{\zeta,\wp}/(t) \tilde{\to} \hat{\mathscr{O}}_{X,\bar{\wp}}$ extends uniquely to an $\hat{\mathscr{O}}_{Z^*,\wp}$-isomorphism $F^*_{\zeta,\wp}$: $R_{\zeta,\wp} \tilde{\to} \hat{\mathscr{O}}_{X,\bar{\wp}}[[t]]$. Here we may identify $\hat{\mathscr{O}}_{Z^*,U} = A[[t]]$, since (Z^*, id) is a thickening of (Z, S_Z); and this induces an identification $\hat{\mathscr{O}}_{Z^*,\wp} = \hat{\mathscr{O}}_{Z,\bar{\wp}}[[t]] = k(\wp)[[t]]$. So letting $X^*_U = \operatorname{Spec} B[[t]]$ and $X^*_\zeta = \operatorname{Spec} R_\zeta$, for every $\wp$ as above we have an isomorphism between the covers that ψ^*_U: $X^*_U \to \operatorname{Spec} \hat{\mathscr{O}}_{Z^*,U}$ and ψ^*_ζ: $X^*_\zeta \to \operatorname{Spec} \hat{\mathscr{O}}_{Z^*,\zeta}$ induce over $k(\wp)[[t]]$. Furthermore, X^*_U is normal since it is the trivial deformation of a smooth curve, and X^*_ζ is normal since each R_ξ is. So by the corollary to Theorem 1, there is a unique normal cover $X^* \to Z^*$ that induces these two covers compatibly with the above identifications.

Note that ψ^*_U: $X^*_U \to \operatorname{Spec} \hat{\mathscr{O}}_{Z^*,U}$ and ψ: $X \to Z$ restrict to the same cover ψ_U: $V \to U$, relative to the above identifications. Similarly, ψ^*_ζ: $X^*_\zeta \to \operatorname{Spec} \hat{\mathscr{O}}_{Z^*,\zeta}$ and ψ restrict to the same cover ψ_ζ: $X_\zeta \to Z_\zeta$, where $Z_\zeta = \operatorname{Spec} \hat{\mathscr{O}}_{Z,\zeta}$ and $X_\zeta = X \times_Z Z_\zeta$. So applying Theorem 1 in the case of $m = 0$ (i.e., no deformation variables), we conclude that there is an identification ϕ: $X \tilde{\to} X^*_{(t)}$ between X and the closed fiber of X^*, with respect to which the restriction of ψ^* to $X^*_{(t)}$ becomes identified with ψ: $X \to Z$. Thus (X^*, ϕ, ψ^*) is a thickening of (X, S_X) relative to ψ, j and is a solution to the given relative thickening problem Θ. This proves surjectivity.

For injectivity, suppose that (X^*, ϕ, ψ^*) and $(\tilde{X}^*, \tilde{\phi}, \tilde{\psi}^*)$ are thickenings of (X, S_X) relative to ψ, j, and that each is a solution to a thickening problem $\Theta = \{(R_\xi, F_\xi, \iota_\xi)\}$ for (X, S_X) relative to ψ, j. Thus we may identify $\hat{\mathscr{O}}_{X^*,\xi}$ and $\hat{\mathscr{O}}_{\tilde{X}^*,\xi}$ with R_ξ as $\hat{\mathscr{O}}_{Z,\psi(\xi)}$-algebras, compatibly with the identifications of X with the closed fibers of X^* and $\tilde{X}^*$. Thus the pullbacks ψ^*_ζ: $X^*_\zeta \to \operatorname{Spec} \hat{\mathscr{O}}_{Z^*,\zeta}$ and $\tilde{\psi}^*_\zeta$: $\tilde{X}^*_\zeta \to \operatorname{Spec} \hat{\mathscr{O}}_{Z^*,\zeta}$ are isomorphic, compatibly with the above identifications. Moreover, so are the pullbacks ψ^*_U: $X^*_U \to \operatorname{Spec} \hat{\mathscr{O}}_{Z^*,U}$ and ψ^*_U: $\tilde{X}^*_U \to \operatorname{Spec} \hat{\mathscr{O}}_{Z^*,U}$, since each is a trivial deformation of $\psi_U: V \to U$ (because X^* and $\tilde{X}^*$ are thickenings of (X, S_X) relative to ψ, j).

Now by the corollary of Theorem 1, $\psi^*: X^* \to Z^*$ is determined by the above data, together with the induced isomorphism between the pullbacks of ψ_U^* and ψ_ζ^* over $k(\wp)[[t]]$, for each prime $\wp$ of $\hat{\mathscr{O}}_{Z,\zeta}$ that is minimal among those containing (t). And the same is true for $\tilde{\psi}^*: \tilde{X}^* \to Z^*$. But these two induced isomorphisms agree modulo (t), since both are given by $F_\zeta = \prod_{\xi \in \psi^{-1}(\zeta)} F_\xi$. So by [Gr2, I, Corollaire 6.2], the two induced isomorphisms agree, compatibly with the above identifications. So the two relative thickenings are isomorphic. This proves injectivity. ∎

Observe that the injectivity part of the above proof actually shows more: namely, any isomorphism between two relative thickening problems lifts uniquely to an isomorphism between the corresponding thickenings.

A relative thickening (X^*, ϕ, ψ^*) of (X, S_X) will be called *G-Galois* if the morphism $\psi^*: X^* \to Z^*$ is G-Galois. Similarly, we will call a relative thickening problem $\Theta = \{(R_\xi, F_\xi, \iota_\xi)\}$ for (X, S_X) *G-Galois* if $X \to Z$ is G-Galois, each $\iota_\zeta: \hat{\mathscr{O}}_{Z^*,\zeta} \to R_\zeta$ is G-Galois (where $R_\zeta = \prod_{\xi \in \psi^{-1}(\zeta)} R_\xi$ and $\iota_\zeta = \prod_{\xi \in \psi^{-1}(\zeta)} \iota_\xi$), and the above G-actions agree over the rings $\hat{\mathscr{O}}_{Z,\zeta}$. Combining the observation of the previous paragraph with Theorem 3, we deduce:

COROLLARY. *Every G-Galois relative thickening problem has a G-Galois solution, which is unique up to isomorphism.*

Namely, by the theorem, the given thickening problem has a solution, and by the above observation, the automorphisms of the thickening problem lift to automorphisms of the solution. Note that the corollary can also be deduced using the Galois version of Theorem 1 (although, of course, the uniqueness proof in Theorem 2 also relies on Theorem 1).

Remark. In the above definitions and results (as in the corollary to Theorem 1), the assumptions of normality and that $m = 1$ (i.e., one deformation variable) are used only to guarantee that the algebras being patched are projective as modules, so that Theorem 1 can be applied. But if [Ha2, Theorem 1(b)] can be generalized to the case of modules that need not be projective, then the same will hold for Theorems 1 and 2 above, and in particular the normality hypothesis in the definition above could be dropped.

In the definition of "relative thickening problem," we are given injections ι_ξ that put an $\hat{\mathscr{O}}_{Z^*,\psi(\xi)}$-algebra structure on each R_ξ. In the proposition below, this requirement is eliminated (thus considering *absolute* thickening problems), in the case that Z^* is a trivial deformation of a smooth curve Z. Uniqueness, however, is no longer asserted. First we prove two lemmas:

LEMMA 1. *Let T be a ring that is complete with respect to an ideal I, and let M be a T-module such that $\bigcap_{n=1}^{\infty} I^n M = (0)$. Let $y_1, \dots, y_s \in M$, and suppose that their images in M/IM generate M/IM as a T/I-module. Then $y_1, \dots, y_s$ generate M as a T-module.*

Proof. Let $m \in M$. By hypothesis, $\bigcap_{n=1}^{\infty} I^n M = (0)$. Hence to show that m is in the T-span of $y_1, \dots, y_s$, it suffices to show that there exist $a_1, \dots, a_s \in T$ such that for all n we have $m - (a_1 y_1 + \cdots a_s y_s) \equiv 0 \pmod{I^n M}$. Since T is complete, to do this it suffices to construct sequences $\{a_{i,n}\}_{n=1}^{\infty}$ in T for $i = 1, \dots, s$ such that

$$m \equiv a_{1,n} y_1 + \cdots + a_{s,n} y_s \pmod{I^n M} \qquad \text{for all } n \tag{$*$}$$

and such that $a_{i,n+1} \equiv a_{i,n} \pmod{I^n}$ for all i, n. We do this by induction.

Namely, for $n = 1$ there exists $a_{1,1}, \dots, a_{s,1} \in T$ such that $m \equiv \sum_{i=1}^{s} a_{i,1} y_i \pmod{IM}$, since $\bar{y}_1, \dots, \bar{y}_s$ generate M/IM as a T/I-module. Given $a_{1,n}, \dots, a_{s,n}$ inductively, there exist elements $t_{1,n}, \dots, t_{N,n} \in I^n$ and $m_{1,n}, \dots, m_{N,n} \in M$ such that $m - \sum_{i=1}^{s} a_{i,n} y_i = \sum_{j=1}^{N} t_{j,n} m_{j,n} \in I^n M$. By the case $n = 1$ applied to $m_{j,n}$, we find that there exist elements $b_{1,j,n}, \dots, b_{s,j,n} \in T$ such that $m_{j,n} \equiv b_{1,j,n} y_1 + \cdots + b_{s,j,n} y_s \pmod{IM}$. Thus the elements $a_{i,n+1} := a_{i,n} + \sum_{j=1}^{N} b_{i,j,n} t_{j,n} \in T$ satisfy the relation $(*)$ with $n+1$ replacing n, and satisfy $a_{i,n+1} \equiv a_{i,n} \pmod{I^n}$. ∎

Using Lemma 1 we obtain

LEMMA 2. *Let k be a field, and let R be a Noetherian complete local domain containing $k[[t]]$, such that t lies in the maximal ideal of R. Let $\bar{\iota}$: $k[[z]] \hookrightarrow R/(t)$ be a finite ring extension. Then there exists an injective $k[[t]]$-algebra homomorphism ι: $k[[z,t]] \hookrightarrow R$ lifting $\bar{\iota}$, and making R a finite $k[[z,t]]$-algebra.*

Proof. Since $\bar{\iota}$: $k[[z]] \hookrightarrow R/(t)$ is finite, the maximal ideal $\bar{\mathbf{m}}$ of $R/(t)$ has the property that $\bar{\iota}^{-1}(\bar{\mathbf{m}}) = (z)$, the maximal ideal of $k[[z]]$. In particular, $\bar{\iota}(z) \in \bar{\mathbf{m}}$. Choose $r \in R$, whose reduction modulo (t) is $\bar{\iota}(z)$.

Since R is complete, and since r, t lie in the maximal ideal $\mathbf{m}$ or R, it follows that for every $f(z,t) = \sum_{i,j} a_{ij} z^i t^j \in k[[z,t]]$, the infinite sum $\sum_{i,j} a_{ij} r^i t^j$ is a well-defined element $f(r,t) \in R$. Moreover, the assignment $f(z,t) \mapsto f(r,t)$ is a well-defined $k[[t]]$-algebra homomorphism ι: $k[[z,t]] \to R$ lifting $\bar{\iota}$.

If $f(z,t) \in k[[z,t]]$ is nonzero, then we may write $f(z,t) = t^n g(z,t)$ for some nonnegative integer n and some $g(z,t) \in k[[z,t]]$ that is not a multiple of t. So the reduction of g modulo (t) is nonzero and hence is not in the kernel of $\bar{\iota}$. Thus $g \notin \ker(\iota)$. Moreover, $\iota(t^n) = t^n \neq 0$. Since R is a domain, $f = t^n g \notin \ker(\iota)$. This proves that ι is injective.

Finally, since R is a Noetherian domain and the ideal tR is nonzero, it follows that $\bigcap_{n=1}^{\infty} t^n R = (0)$ [AM, Corollary 10.18]. So applying Lemma 1 with $T = k[[z,t]]$, $I = (t)$, and $M = R$, and using that $R/(t)$ is finite over $k[[z]] = T/(t)$, we have that R is finite over $k[[z,t]]$. ∎

Applying Lemma 2 to Theorem 2, we obtain:

PROPOSITION 1. *Let Θ be a thickening problem for (X, S_X), where X is a connected reduced projective k-curve and $S_X \subset X$ is a nonempty finite closed subset containing the singular locus of X. Let $\psi\colon X \to Z$ be a finite generically separable morphism to a smooth projective k-curve Z, such that $S_X = \psi^{-1}(S_Z)$ for some finite subset $S_Z \subset Z$. Let $Z^* = Z \times_k k[[t]]$. Then for some solution (X^*, ϕ) to the (absolute) thickening problem Θ, there is a finite morphism $\psi^*\colon X^* \to Z^*$ whose closed fiber is $\psi \circ \phi^{-1}$.*

Proof. Fix any $\zeta \in S_Z$, and let z be a local parameter on Z at ζ. Then for each $\xi \in S_\zeta = \psi^{-1}(\zeta)$, the restriction of ψ to Spec $\hat{\mathcal{O}}_{X,\xi}$ corresponds to an inclusion $\psi_\xi\colon k[[z]] \hookrightarrow \hat{\mathcal{O}}_{X,\xi}$, and hence (by composing with $F_\xi^{-1}\colon \hat{\mathcal{O}}_{X,\xi} \xrightarrow{\sim} R_\xi/(t)$) an inclusion $\bar{\iota}_\xi\colon k[[z]] \hookrightarrow R_\xi/(t)$. Since ψ is finite, this inclusion makes $R_\xi/(t)$ into a finite $k[[z]]$-algebra. Applying Lemma 2, this lifts to an inclusion $\iota_\xi\colon k[[z,t]] \hookrightarrow R_\xi$ that makes R_ξ a finite $k[[z,t]]$-algebra. Moreover, we have the equality $F_\xi \circ \bar{\iota}_\xi = \psi_\xi\colon \hat{\mathcal{O}}_{Z,\zeta} \hookrightarrow \hat{\mathcal{O}}_{X,\xi}$. So by the above Theorem 2, the conclusion follows. ∎

Using the proposition, we obtain the desired existence theorem for solutions to absolute thickening problems:

THEOREM 3. *Every thickening problem has a solution.*

Proof. Let Θ be a thickening problem for (X, S_X), where X is a connected reduced projective k-curve and $S_X \subset S$ is a nonempty finite closed subset containing the singular locus of X. If there is a finite generically separable morphism $\psi\colon X \to \mathbf{P}_k^1$ such that $S_X = \psi^{-1}(\infty)$, then by applying the above proposition to $Z = \mathbf{P}^1$ and $S_Z = \{\infty\}$, we conclude that there is a solution (X^*, ϕ) to Θ. Thus it suffices to show that such a ψ exists.

Let $\tilde{X}$ be the normalization of X and let $S_{\tilde{X}} \subset \tilde{X}$ be the inverse image of $S_X \subset X$ in $\tilde{X}$. By Riemann–Roch for smooth projective curves, if m is a sufficiently large integer, then there is a rational function f on $\tilde{X}$ that is regular away from $S_{\tilde{X}}$, and which has a pole of order m at each point of $S_{\tilde{X}}$. Here f corresponds to a morphism $\tilde{\psi}_0\colon \tilde{X} \to \mathbf{P}^1$ such that $\tilde{\psi}_0^{-1}(\infty) = S_{\tilde{X}}$. Moreover, by taking m prime to p, we may assume that $\tilde{\psi}_0$ is generically separable. Now the rational function $1/f \in K(\tilde{X})$ has a zero at each point of $S_{\tilde{X}}$, and so by [Se, IV, Section 1.2(4)] there is an integer N such that if $n \geq N$, then $1/f^n$ descends to an element in the maximal ideal of the local ring of X at each of the points of S_X. (Actually, Serre assumes that the base field k is algebraically closed, but that assumption was not used in obtaining [Se, IV, Section 1.2(4)].) Meanwhile, the restriction of $1/f^n$ to $\tilde{X} - S_{\tilde{X}}$ descends to a rational function on the smooth curve $\tilde{X} - S_{\tilde{X}} \approx X - S_X$, corresponding to a morphism to $\mathbf{P}^1$. So the morphism $\tilde{\psi}\colon \tilde{X} \to \mathbf{P}^1$ corresponding to f^n, which satisfies $\tilde{\psi}^{-1}(\infty) = S_{\tilde{X}}$, descends to a morphism $\psi\colon X \to \mathbf{P}^1$ such that $\psi^{-1}(\infty) = S_X$. If n is chosen prime to p, then $\tilde{\psi}$ is generically separable (since $\tilde{\psi}_0$ is) and hence so is ψ. ▮

Remark. In the absolute case (i.e., Theorem 3), unlike in the relative case (i.e., Theorem 2), the solution need not be unique up to isomorphism. For example, let X be the elliptic curve given in affine coordinates by $y^2 = x(x^2 - 1)$, and let $S = \{\xi\}$, where ξ is the point where $x = y = 0$. Thus y is a local uniformizer at ξ, and we may identify $\hat{\mathscr{O}}_{X,\xi}$ with $k[[y]]$. Let $R_\xi = k[[y, t]]$, and let $F_\xi\colon R_\xi/(t) \xrightarrow{\sim} \hat{\mathscr{O}}_{X,\xi}$ be the obvious map. Thus (R_ξ, F_ξ) defines a thickening problem Θ for (X, S). The trivial thickening of (X, S) is given by the space $X^* = X \times_k k[[t]]$, together with the obvious isomorphism of X with the closed fiber of X^*. Moreover, this is a solution to the thickening problem Θ. But another solution to Θ is given by the thickening $(\tilde{X}^*, \tilde{\phi})$, where $\tilde{X}^*$ is the elliptic curve over $k[[t]]$ that is given in affine coordinates by $y^2 = (x - t)(x^2 - 1)$, and where $\tilde{\phi}$ is the obvious isomorphism of X with the closed fiber of $\tilde{X}^*$. But these two solutions are not isomorphic, because the generic fibers of X^* and $\tilde{X}^*$ are elliptic curves over $k((t))$ that have different j-invariants.

3. THICKENING AND DEFORMING COVERS

In order to use formal geometry to construct covers with desired Galois groups and inertia groups, one wants to begin with a degenerate cover $X \to Z$ of a possibly reducible base curve Z, having the desired Galois group. The next step is to thicken both X and Z to obtain a cover of irreducible normal curves over $k[[t]]$. Theorem 4 below shows that such a thickening of covers is possible, and its proof relies on the results of the previous sections. Then Theorem 5 and its corollaries provide the set-up so that actual constructions can be performed (as will be done in the following section).

Below we fix a field k, and consider k-curves Z that are reduced and connected but not necessarily irreducible. Given a cover $\psi\colon X \to Z$, we consider the existence of covers $\psi^*\colon X^* \to Z^*$ of normal $k[[t]]$-curves whose closed fiber is $\psi\colon X \to Z$, with specified behavior near the singular points.

More explicitly, by a *thickening problem for covers*, over k, we will mean the following data:

(i) A cover $\psi\colon X \to Z$ of geometrically connected reduced projective k-curves, together with a finite closed set $S \subset Z$ containing the singular locus of Z.

(ii) For every $\zeta \in S$, a Noetherian normal complete local domain R_ζ containing $k[[t]]$, and whose maximal ideal contains t, together with a finite generically separable R_ζ-algebra A_ζ.

(iii) For every $\zeta \in S$, a pair of k-algebra isomorphisms $F_\zeta\colon R_\zeta/(t) \xrightarrow{\sim} \hat{\mathscr{O}}_{Z,\zeta}$ and $E_\zeta\colon A_\zeta/(t) \xrightarrow{\sim} \hat{\mathscr{O}}_{X,\zeta} := \prod_{\xi \in \psi^{-1}(\zeta)} \hat{\mathscr{O}}_{X,\xi}$ that are compatible with the given inclusions $\iota_\zeta\colon R_\zeta \hookrightarrow A_\zeta$ and $\hat{\mathscr{O}}_{Z,\zeta} \hookrightarrow \hat{\mathscr{O}}_{X,\zeta}$. That is, the following induced diagram commutes:

$$\begin{tikzcd}
A_\zeta(t) \arrow[r, "\sim"] & \hat{\mathscr{O}}_{X,\zeta} \\
R_\zeta/(t) \arrow[r, "\sim"] \arrow[u, hook, "\iota_\zeta"] & \hat{\mathscr{O}}_{Z,\zeta} \arrow[u, hook]
\end{tikzcd}$$

Given such a thickening problem for covers, write $Z' = Z - S$ and $X' = X - \psi^{-1}(S)$. A *solution* to such a thickening problem for covers consists of a cover $\psi^*\colon X^* \to Z^*$ of projective normal $k[[t]]$-covers whose closed fiber is isomorphic to $X \to Z$; whose pullback $\psi'^*\colon X'^* \to Z'^* := \operatorname{Spec} \hat{\mathscr{O}}_{Z^*, Z'}$ is a trivial deformation of the restriction $\psi'\colon X' \to Z'$; and whose pullbacks over the complete local rings at the points of S are given by $R_\zeta \subset A_\zeta$, compatibly with the above isomorphisms.

Similarly, for any finite group G we may consider the corresponding notions of a *G-Galois thickening problem for covers*, and the *solution* to such a problem. Namely, for the former notion we require that $\psi\colon X \to Z$ and $R_\zeta \subset A_\zeta$ are each G-Galois, compatibly with F_ζ. Similarly, for the latter notion, we require that $\psi\colon X^* \to Z^*$ be G-Galois, and that this G-Galois action induce the actions on the closed fiber and on the extensions $R_\zeta \subset A_\zeta$.

Combining Theorems 2 and 3, we then obtain the following:

THEOREM 4. *Every thickening problem for covers has a solution, as does every thickening problem for G-Galois covers.*

Proof. Suppose we are given a thickening problem for covers as above. The data $\{(R_\zeta, F_\zeta)\}$ define a thickening problem for the curve Z, and by Theorem 3 there is a solution (Z^*, ϕ). Thus in particular there is an inclusion $j\colon Z \hookrightarrow Z^*$ that identifies Z with the closed fiber of Z^*.

Meanwhile, the decomposition $\hat{\mathscr{O}}_{X,\zeta} = \prod_{\xi \in \psi^{-1}(\zeta)} \hat{\mathscr{O}}_{X,\xi}$ lifts to a decomposition $A_\zeta = \prod_{\xi \in \psi^{-1}(\zeta)} A_\xi$, where $A_\xi := E_\zeta^{-1}(\hat{\mathscr{O}}_{X,\xi})$ is a finitely generically separable R_ξ-algebra. For each $\xi \in X$ over $\zeta \in S$, we have that E_ζ restricts to an isomorphism $E_\xi\colon A_\xi/(t) \xrightarrow{\sim} \hat{\mathscr{O}}_{X,\xi}$, and ι_ζ restricts to an inclusion of $k[[t]]$-algebras $\iota_\xi\colon \hat{\mathscr{O}}_{Z,\zeta} \hookrightarrow A_\xi$. By (iii), E_ξ and ι_ξ are compatible, i.e., the reduction $\bar{\iota}_\xi\colon \hat{\mathscr{O}}_{Z,\zeta} \hookrightarrow R_\xi/(t)$ modulo (t) has the property that $E_\xi \circ \bar{\iota}_\xi\colon \hat{\mathscr{O}}_{Z,\zeta} \hookrightarrow \hat{\mathscr{O}}_{X,\xi}$ is the map induced by completely localizing ψ at ξ. Thus the data $\{(A_\xi, E_\xi, \iota_\xi)\}$ define a thickening problem for $(X, \psi^{-1}(S))$ relative to ψ and j. By Theorem 2, there is a solution (X^*, ϕ, ψ^*) to this relative thickening problem. It is then immediate that the cover $\psi^*\colon X^* \to Z^*$ is a solution to the given thickening problems for covers.

The G-Galois case is similar, using the corollary to Theorem 2. ∎

Remark. The above proof shows more: namely, that once we are given Z^* as above, the cover $\psi\colon X^* \to Z^*$ is unique up to isomorphism. This is because of the uniqueness assertion in Theorem 2. But the choice of Z^* is not unique, since there is no uniqueness in Theorem 3.

Let N_k be the set of natural numbers such that k contains a primitive nth root of unity. For each $n \in N_k$, we may choose a primitive nth root of unity $\omega_n \in k$ such that $\omega_{mn}^m = \omega_n$ for all $mn, n \in N_k$. Now consider a G-Galois cover $\psi\colon X \to Z$ of smooth connected k-curves such that each ramification point is k-rational and each ramification index lies in N_k. Thus in particular, the cover is tamely ramified. Let $\xi \in X(k)$, lying over $\zeta \in Z(k)$, be a point with ramification index n. The corresponding extension $\hat{\mathscr{O}}_{Z,\zeta} \subset \hat{\mathscr{O}}_{X,\xi}$ of complete local rings is Galois, and its Galois group, which is cyclic of order n, is the inertia group of $X \to Z$ at ξ. Let z, x be uniformizers at ζ, ξ respectively. Then by Kummer theory this extension of complete local rings is given by $x^n = az$ for some $a \in k^*$, and the automorphism $c_\xi\colon x \mapsto \omega_n x$ generates the inertia group. Moreover, the automorphism c_ξ is independent of the choices of x, z, and a; and if k is algebraically closed then we may take $a = 1$. As in [St1], we will call this element $c_\xi \in G$ the *canonical generator* of the inertia group at ξ. This is determined by the point ζ up to conjugacy. If the branch points of $X \to Z$ are given with an ordering, say $\zeta_1, \dots, \zeta_r$, then we say that the cover has *description* $(c_1, \dots, c_r)$, where c_j is a canonical generator of inertia at a point over ζ_j, and where each c_j is determined up to conjugacy. (This notion is related to that of "branch cycle description" of covers over the complex numbers [Fr], but is weaker, since the entries of a branch cycle description are determined up to *uniform* conjugacy.)

Consider a G-Galois cover $\psi\colon X \to Z$ of semistable k-curves, i.e., that every singularity of X or of Z is an ordinary double point rational over k. Let S_Z be the set of double points of Z, and suppose that the points of

$S_X \coloneqq \psi^{-1}(S_Z)$ are k-rational. Let $\nu\colon \tilde{Z} \to Z$ be the normalization of Z, and let $\hat{\nu}\colon \tilde{X} \to X$ be the pullback of ν under $\psi\colon X \to Z$. Thus for each point $\xi \in S_X$, the inverse image $\hat{\nu}^{-1}(\xi) \subset \tilde{X}$ consists of two points. Suppose that $\tilde{X}$ is smooth and that the (possibly disconnected) G-Galois cover $\tilde{\psi}\colon \tilde{X} \to \tilde{Z}$ is tamely ramified over $S_{\tilde{Z}} \coloneqq \nu^{-1}(S_Z)$. If, for each $\xi \in S_X$, the canonical generators of inertia at the two points of $\hat{\nu}^{-1}(\xi)$ are inverses in G, then we will call the G-Galois cover $X \to Z$ *admissible*.

Thus if $X \to Z$ as above is an admissible G-Galois cover, then the complete local ring at any point $\zeta \in S_Z$ is of the form $k[[u, v]]/(uv)$. Moreover, if $\xi \in S_X$ lies over ζ, then the complete local ring at ξ is of the form $k[[x, y]]/(xy)$, and the inclusion $\hat{\mathscr{O}}_{Z,\zeta} \subset \hat{\mathscr{O}}_{X,\xi}$ is given by $x^n = au$, $y^n = bv$, where n is the ramification index at the point and where $a, b \in k^*$. At the two points of $\tilde{X}$ lying over ξ, the canonical generators of inertia are elements g and g^{-1} of G, where g acts on $\hat{\mathscr{O}}_{X,\xi}$ by $g(x) = \omega_n x$, $g(y) = \omega_n^{-1} y$.

Let $\psi\colon X \to Z$ be an admissible G-Galois cover, and let $X' = X - S_X$ and $Z' = Z - S_Z$. Let $\psi^*\colon X' \to Z^*$ be a G-Galois cover of normal $k[[t]]$-curves whose closed fiber is $\psi\colon X \to Z$, and whose pullback $\psi'^*\colon X'^* \to Z'^* \coloneqq \operatorname{Spec} \hat{\mathscr{O}}_{Z^*, Z'}$ is a trivial deformation of the restriction $\psi'\colon X' \to Z'$. We will call $\psi^*\colon X^* \to Z^*$ an *admissible thickening* of $\psi\colon X \to Z$ if for each $\zeta \in S_Z$ and $\xi \in \psi^{-1}(\zeta)$, the extension of complete local rings $\hat{\mathscr{O}}_{Z^*,\zeta} \subset \hat{\mathscr{O}}_{X^*,\xi}$ is given by

$$k[[t, u, v]]/(uv - t^n) \subset k[[t, x, y]]/(xy - abt),$$

where as above $x^n = au$, $y^n = bv$ $(a, b \in k^*)$; n is the ramification index over ζ; and the canonical generator $g \in G$ (as above) acts by $g(x) = \omega_n x$, $g(y) = \omega_n^{-1} y$. To indicate the dependence on ζ, we sometimes write $a = a_\zeta$, $b = b_\zeta$, $n = n_\zeta$.

In these terms, we obtain the following corollary of Theorem 4:

COROLLARY. *Every admissible cover has an admissible thickening.*

Proof. Let $X \to Z$ be an admissible G-Galois cover as above, and preserve the above notation. Thus at each $\zeta \in S_Z$ we have a ramification index $n_\zeta \in N_k$ and elements $a_\zeta, b_\zeta \in k^*$. For each $\zeta \in S_Z$ choose a point of $\tilde{X}$ over ζ, let $g \in G$ be the canonical generator of inertia at that point, of order n_ζ, and let $I_\zeta \subset G$ be the subgroup generated by g. Let $R_\zeta = k[[t, u, v]]/(uv - t^{n_\zeta})$, and let $A_\zeta = \operatorname{Ind}_{I_\zeta}^G k[[t, x, y]]/(xy - a_\zeta b_\zeta t)$. Thus we have compatible isomorphisms $F_\zeta\colon R_\zeta/(t) \xrightarrow{\sim} \hat{\mathscr{O}}_{Z,\zeta}$ and $E_\zeta\colon A_\zeta/(t) \xrightarrow{\sim} \hat{\mathscr{O}}_{X,\zeta} \coloneqq \prod_{\xi \in \psi^{-1}(\zeta)} \hat{\mathscr{O}}_{X,\xi}$. Thus we have a thickening problem for G-Galois covers, and by Theorem 4 there is a solution. This solution is then an admissible thickening of $X \to Z$. ∎

Remark. Theorems 3 and 4 can also be viewed from the perspective of rigid geometry, and in these terms are related to [Sa1] and [Sa2].

Florian Pop [Po2] has defined a field k to be *large* if it has the property that every smooth geometrically irreducible k-variety with a k-point has a dense set of k-points. This includes many familiar classes of fields, e.g., algebraically closed fields, fields that are complete with respect to a discrete valuation, PAC fields, the fields $\mathbf{Q}^{tr}$ and $\mathbf{Q}^{tp}$ of totally real and totally p-adic algebraic numbers, etc. Furthermore, every algebraic extension of a large field is large. For such fields, it is possible to specialize the solution to a thickening problem for covers, and one obtains the following consequence of Theorem 4:

THEOREM 5. *Let k be a large field, and consider a G-Galois thickening problem for k-covers given by $\psi: X \to Z$ and by extensions $\iota_\zeta: R_\zeta \hookrightarrow A_\zeta$, for $\zeta \in S \subset Z$. Suppose that away from S the branch locus of $X \to Z$ consists of m distinct k-points, and that for $\zeta \in S$ the branch locus of the generic fiber of $\iota_\zeta: R_\zeta \hookrightarrow A_\zeta$ consists of m_ζ distinct $k((t))$-points. Let γ be the arithmetic genus of Z.*

Then there is a smooth projective k-curve $Z^\#$ of genus γ, and a smooth connected G-Galois branched cover of curves $X^\# \to Z^\#$ whose branch locus consists of $m + \sum_{\zeta \in S} m_\zeta$ distinct k-points. Moreover, the inertia groups at these points (and, in the tame case, the description) may be taken to be equal to those arising from ψ and the ι_ζ's as above.

Proof. By Theorem 4, there is a solution $\psi^*: X^* \to Z^*$ to the given G-Galois thickening problem. The scheme Z^* is flat over the discrete valuation ring $k[[t]]$, since its structure sheaf $\mathscr{O}_{Z^*}$ is torsion-free over $k[[t]]$. So $\mathscr{O}_{Z^*}$ is locally free over $k[[t]]$. Here the morphism ψ^*, as well as the G-action and the branch points, are all defined over $k[[t]]$; and the inertia groups away from S (and the description, in the tame case) agree with those of ψ, since ψ^* is a trivial deformation of ψ away from S. Since the structure is of finite type, there is a $k[t]$-subalgebra $T \subset k[[t]]$ of finite type, over which the covering morphism, G-Galois action, and branch points are defined, and such that the structure sheaf of the base space is locally free over T. Let $V = \operatorname{Spec} T$. Thus V is a k-scheme whose function field $K(V)$ is contained in $k((t))$. Since k is algebraically closed in $k((t))$, it follows that $K(V)$ and $\bar{k}$ are linearly disjoint over k. Hence V is a geometrically irreducible k-variety.

Thus we have a G-Galois cover of projective V-curves $X_V \to Z_V$ whose branch points are V-rational, which induces $X^* \to Z^*$ via pullback by $\operatorname{Spec} k[[t]] \to V$, and such that $Z_V \to V$ is flat. The composition $T \hookrightarrow k[[t]] \twoheadrightarrow k$ (where $t \mapsto 0$ under the second map) corresponds to a k-valued point τ of V, and the fiber over this point is the original G-Galois cover $X \to Z$.

So there is a dense open subset $U_1 \subset V$ such that for all closed points $\mu \in U_1$, the corresponding fiber $X_\mu \to Z_\mu$ of $X_V \to Z_V$ is a cover with branch points defined over $k(\mu)$, and with inertia groups (and description, in the tame case) as desired. Also, by [Ha4, Proposition 5] (or by the Bertini–Noether theorem [FJ, Proposition 9.29]), there is a dense open subset $U_2 \subset V$ such that for all closed points $\mu \in U_2$, the fiber $X_\mu \to Z_\mu$ is a morphism of geometrically irreducible varieties over $k(\mu)$. So for $\mu \in U = U_1 \cap U_2$, the fiber over μ has both properties. Moreover, since $Z_V \to V$ is flat, it follows by [Ht, III, Corollary 9.10] that the arithmetic genus of the fiber Z_μ is equal to that of $Z_\tau = Z$, viz. γ.

Since the field k is large, and since $V(k) \neq \varnothing$, it follows that the set of k-points of V is dense. So the dense open set U contains a k-point μ, and the fiber over μ is then as desired. ∎

For any semistable k-curve U, let $\pi_A^{\mathrm{adm}}(U)$ be the set of (isomorphism classes of) finite groups G such that there exists a connected G-Galois admissible cover of U. In the case that U is smooth, this is just the set $\pi_A(U)$ of finite Galois groups of connected étale covers of U, i.e., the continuous finite quotients of $\pi_1(U)$. Also, for any integer γ, by $\pi_A(\gamma)$ we denote the set of groups G for which there is a dense open subset M_G in the moduli space $\mathscr{M}_\gamma$ of curves of genus γ, such that $G \in \pi_A(U)$ for all k-curves U in M_G. The above two results then yield the following:

COROLLARY 1. *Let k be a large field, and let Z be a connected semistable k-curve of arithmetic genus γ, with singular locus S. Let $\Delta \subset Z - S$ be a (possibly empty) finite set of closed points.*

(a) *Then there is a smooth projective k-curve $Z^\#$ and a finite subset $\Delta^\# \subset Z^\#$ of k-points with $|\Delta^\#| = |\Delta|$, such that $\pi_A^{\mathrm{adm}}(Z - \Delta) \subset \pi_A(Z^\# - \Delta^\#)$.*

(b) *Moreover, given a connected G-Galois admissible cover $X \to Z$, there is a smooth connected G-Galois cover $X^\# \to Z^\#$ with branch locus $\Delta^\#$ as above, such that the two covers have the same inertia groups and, in the tame case, the same description.*

Proof. Let $X \to Z$ be as in (b), where possibly $X = Z$; and for $\zeta \in S$ let $n_\zeta \in N_k$ and $a_\zeta, b_\zeta \in k^*$ be as before. By the corollary to Theorem 4, this cover has an admissible thickening $X^* \to Z^*$, viz. a solution to the thickening problem for G-Galois covers given by the extensions $R_\zeta \subset A_\zeta$ for $\zeta \in S$, where $R_\zeta = k[[t,u,v]]/(uv - t^{n_\zeta})$ and $A_\zeta = \mathrm{Ind}_{\langle g_\zeta \rangle}^G k[[t,x,y]]/(xy - a_\zeta b_\zeta t)$. The extension $R_\zeta \subset A_\zeta$ is ramified only over the maximal ideal of R_ζ, and in particular is unramified over the general fiber (i.e., away from $t = 0$). So by Theorem 5 there is a smooth connected G-Galois cover $X^{\#} \to Z^{\#}$ branched at a set $\Delta^{\#}$ of cardinality equal to that of Δ, with the desired properties. This proves (a) and (b). ∎

In fact, more is true when k is algebraically closed (and perhaps in general for large fields). In the case where $\Delta \neq \emptyset$ in the above corollary, we also have that $\Delta^{\#} \neq \emptyset$, and so the curve $Z^{\#} - \Delta^{\#}$ is a smooth affine curve. So $\pi_A(Z^{\#} - \Delta^{\#})$ depends only on the genus of $Z^{\#}$ and on $|\Delta^{\#}|$, by the Abhyankar Conjecture [Ha5]. Thus the conclusion of part (a) holds for *all* smooth projective k-curves $Z^{\#}$ and *all* subsets $\Delta^{\#} \subset Z^{\#}$ with $|\Delta^{\#}| = |\Delta|$.

On the other hand, in the case where $\Delta = \emptyset$, the conclusion of part (a) of the corollary does *not* hold in general for arbitrary $Z^{\#}$ in characteristic p, even if $k = \bar{k}$, since π_A can vary among projective curves of a given genus. For example, let Z consist of two copies of $\mathbf{P}^1$ meeting at two points, so that the arithmetic genus is 1. The curve Z has an unramified connected admissible cover $X \to Z$ that is cyclic of order $p = \operatorname{char} k$, but supersingular elliptic curves $Z^{\#}$ do not have any unramified p-cyclic covers. Nevertheless, we have the following result in that case:

COROLLARY 2. *If Z is a connected semistable projected curve of arithmetic genus γ over a large field k, then $\pi_A^{\mathrm{adm}}(Z) \subset \pi_A(\gamma)$.*

Proof. This follows from the above result via [St1, Proposition 4.2], which asserts that if $G \in \pi_A$ of a smooth projective curve of genus γ, then $G \in \pi_A$ of all curves that correspond to points in some dense open subset of $\mathscr{M}_\gamma$. ∎

4. APPLICATIONS TO FUNDAMENTAL GROUPS

In this section we consider applications of the above results to fundamental groups of curves in characteristic p, in both the tame and wild cases. Our goal is to construct covers with given Galois group, branch locus, and inertia groups, by patching together simpler (known) covers. In doing so, we will use the results of the previous sections on patching and thickening problems, together with specialization. To specialize we will need to assume that the base field k is large (e.g., algebraically closed), as in Theorem 5 of Section 3. Using this approach, we provide (in Theorem 6) a simple proof of a key step in Raynaud's proof of the Abhyanker Conjecture for $\mathbf{A}^1$, by patching together covers of the line at wildly ramified branch points. We also show how to patch together covers at tamely ramified points, thereby obtaining Theorem 7 on the construction of covers of curves of a given genus. This leads to several examples of explicit applications, which appear in corollaries to that result.

We first consider the wildly ramified case. Here, for example, we will patch together two (or more) étale covers of the affine line, by suitably identifying the ramification points over infinity. In the inductive step, we will consider copies of the u-line and v-line over k, and will construct a new cover with desired properties (e.g., larger inertia) by solving an appropriate patching problem. We begin by recalling some basic facts.

Let k be a field of characteristic p, let $X = \operatorname{Spec} R$ be an affine scheme over k, and let P be a nontrivial finite p-group. Then P contains a central cyclic subgroup A of order p; let $\bar{P} = P/A$. Every A-Galois étale cover of X is given by an Artin–Schreier equation $y^p - y = r$, for some $r \in R$, and the set of isomorphism classes of such covers may be identified with the group $\operatorname{Hom}(\pi_1(X), A)$. Here the group operations is given by $(\alpha \cdot \beta)(g) = \alpha(g)\beta(g)$, or on the ring level by adding the elements r in the Artin–Schreier equations.

If we pick a $\bar{P}$-Galois étale cover and then consider the isomorphism classes of P-Galois étale covers that dominate it, then the resulting set is a principal homogenous space for the above group. Namely, the given $\bar{P}$-Galois cover corresponds to an element $\bar{\gamma}$ in the set $\operatorname{Hom}(\pi_1(X), \bar{P})$, and the group $\operatorname{Hom}(\pi_1(X), A)$ acts on the fiber $\operatorname{Hom}_{\bar{\gamma}}(\pi_1(X), P)$ of $\operatorname{Hom}(\pi_1(X), P) \to \operatorname{Hom}(\pi_1(X), \bar{P})$ over $\bar{\gamma}$ by $(\alpha \cdot \gamma)(g) = \alpha(g)\gamma(g)$. Here this action is transitive, since if γ, γ' are in the fiber $\operatorname{Hom}_{\bar{\gamma}}(\pi_1(X), P)$, then $g \mapsto \gamma(g)\gamma'(g)^{-1}$ lies in $\operatorname{Hom}(\pi_1(X), A)$, by centrality of A.

The action of $\operatorname{Hom}(\pi_1(X), A)$ on $\operatorname{Hom}_{\bar{\gamma}}(\pi_1(X), P)$ can be expressed on the ring level as follows. Choose a sequence of normal subgroups $N_0 \supset N_1 \supset \cdots \supset N_m$ of P, where $(N_{i+1}: N_i) = p$, $N_0 = P$, $N_{m-1} = A$, and $N_m = 1$. Let $P^i = P/N_i$. Thus $P^0 = 1$, $P^{m-1} = \bar{P}$, and $P^m = P$. Any P-Galois cover may then be expressed by a sequence of Artin–Schreier equations $z_1^p - z_1 = r_1, \dots, z_m^p - z_m = r_m$, where for $0 \le i \le m$ the equations $z_1^p - z_1 = r_1, \dots, z_i^p - z_i = r_i$ define a P^i-Galois cover $X_i = \operatorname{Spec} R_i \to X$ and where each $r_j \in R_{j-1}$. Given a P-Galois étale cover corresponding to the sequence $(r_1, \dots, r_{m-1}, r_m)$ and given an A-Galois cover given by $y^p - y = r$ with $r \in R = R_0$, the latter cover acts by taking the former to the P-Galois cover corresponding to the sequence $(r_1, \dots, r_{m-1}, r_m + r)$.

If $Q \subset P$ and $Y \to X$ is a Q-Galois étale cover, then there is an induced P-Galois étale cover $\operatorname{Ind}_Q^P Y \to X$, where $\operatorname{Ind}_Q^P Y$ is a disjoint union of copies of Y, indexed by the left cosets of Q in P. On the ring level, if $Y = \operatorname{Spec} T$, then $R \subset T$ is given by an n-tuple $(r_1, \dots, r_n)$ as above, where $\#Q = p^n$. The corresponding induced P-Galois ring extension $R \subset \operatorname{Ind}_Q^P T$ is given by $(r_1', \dots, r_m')$, where $\#P = p^m$ and where the entries r_i' are given by entries of r_j interspersed with entries of 0, corresponding to values of i for which $Q \cap P^i = Q \cap P^{i+1}$ (with notation as above).

To patch together covers of the u-line and of the v-line over a field k, we will use the cone given by $uv = t^2$, whose fiber modulo t consists of a union of the two given lines crossing normally. The tangent plane to the cone along the line L: $(u = v = t)$ is given by $u + v = 2t$, and so the complement of L is an affine open subset of the cone, obtained by inverting $u + v - 2t$. If the given covers have p-group ramification at the origin, then the patched cover will be taken to have ramification over the line L. To accomplish the patching via the results of the previous section, we work over the complete local rings at the origin, and use the following lemma:

LEMMA. *Let k be a field, $R = k[[t,u,v]]/(uv - t^2)$, and $R' = R[1/(u + v - 2t)]$. Let P be a p-group, and consider P-Galois étale ring extensions $k((u)) = R'/(v) \subset \Omega_1$ and $k((v)) = R'/(u) \subset \Omega_2$. Then there is a P-Galois étale extension $R' \subset \Omega$ of domains whose reductions modulo (v) and (u) are $k((u)) \subset \Omega_1$ and $k((v)) \subset \Omega_2$, respectively, such that the normalization of R in Ω is totally ramified over the locus $(u = v = t)$.*

Proof. Let $\hat{X} = \operatorname{Spec} \Omega_1$ and $\hat{Y} = \operatorname{Spec} \Omega_2$. These are (possibly disconnected) P-Galois covers of $\hat{U} := \operatorname{Spec} k((u))$ and $\hat{V} := \operatorname{Spec} k((v))$, respectively. In addition, let $D = \operatorname{Spec} R$ and $D' = \operatorname{Spec} R'$.

We proceed by induction on the order of P. Let $A \subset P$ be a central cyclic subgroup of order p, and $\bar{P} = P/A$. Let $\bar{X} = \hat{X}/A \to U$ and $\bar{Y} = \hat{Y}/A \to V$ be the $\bar{P}$-Galois covers dominated by the P-Galois covers $\hat{X}$ and $\hat{Y}$. Let $\bar{P}_i = P_i \cap \bar{P}$ for $i = 1, 2$. Since $\#\bar{P} < \#P$, the inductive hypothesis applies to the quotient situation. So there is a connected $\bar{P}$-Galois étale cover $\bar{C}' \to D'$ whose fiber over $t = 0$ has the property that its pullbacks to $k((u))$ and $k((v))$ agree with those of $\bar{X}$ and $\bar{Y}$, and such that the normalization $\bar{C}$ of D in $\bar{C}'$ is totally ramified over $(u = v = t)$.

Since $\operatorname{cd}_p(R') \le 1$ [AGV, X, Théorème 5.1], $\bar{C}' \to D'$ is dominated by a P-Galois étale cover $C'' \to D'$. Thus the fibers of C'' and $\hat{X}$ over $\hat{U}$ are P-Galois covers whose quotients by A yield the same $\bar{P}$-Galois cover. Since $\operatorname{Hom}(\pi_1(\operatorname{Spec} k((u))), A)$ acts transitively on the fibers of $\operatorname{Hom}(\pi_1(\operatorname{Spec} k((u))), P) \to \operatorname{Hom}(\pi_1(\operatorname{Spec} k((u))), \bar{P})$, there is an A-Galois cover $z^p - z = r$ that carries C'' to $\hat{X}$; here $r \in k((u))$. Similarly, there is an A-Galois cover $z^p - z = s$ carrying C'' to $\hat{Y}$, where $s \in k((v))$.

For $n \gg 0$, we may write $r = f/u^n$ and $s = g/v^n$, where $f \in uk[[u]]$ and $g \in vk[[v]]$. Choose such an n, and an element $h \in tk[[t]]$, and consider the A-Galois cover $E' \to D'$ given by $z^p - z = e$, where $e = (f + g + h)/(u + v - 2t)^n \in R'$. Let $C' \to D'$ be the P-Galois étale cover obtained by letting this A-Galois cover act on $C'' \to D'$, and let $C \to D$ be the normalization of D in C'. (Thus if $C'' \to D'$ corresponds to the m-tuple $(r_1, \dots, r_m)$ as above, then $C' \to D'$ corresponds to $(r_1, \dots, r_{m-1}, r_m + e)$.)

Since $e \equiv r \pmod{v}$ and $e \equiv s \pmod{u}$ in R, it follows that the fiber of $C' \to D'$ over $(t = 0)$ has the property that its pullbacks to $k((u))$ and $k((v))$ agree with those of $\hat{X}$ and $\hat{Y}$, respectively.

For $n \gg 0$ and for a generic choice of $h \in tk[[t]]$, the P-Galois cover $C \to D$ is totally ramified over $(u + v - 2t)$, i.e., over $(u = v = t)$, since $\overline{C} = C/A \to D$ is. Hence C and C' are irreducible. Writing $C' = \operatorname{Spec} \Omega$, the extension $R' \subset \Omega$ is then as desired. ∎

Combining the above with the results of Section 3, we obtain the following proof of [Ra, Théorème 2.2.3], which was shown in Section 5 of [Ra] using the machinery of Sections 3 and 4 of that paper. The proof here uses the machinery of formal geometry as above, rather than rigid geometry as in [Ra]. In applying Theorem 5 above, the proof here uses a strategy that is related to that of an earlier, unpublished version of [Ra] (although that used rigid methods). Furthermore, we state the result a bit more generally than in [Ra], viz. for *large* fields rather than just for algebraically closed fields. This is possible since Theorem 5 above required only that the base field k be large. In the framework of [Ra], this more general conclusion can also be obtained, since the "large" hypothesis allows specialization to k as in Theorem 5.

THEOREM 6. *Let G be a finite group generated by subgroups $G_1, \ldots, G_n$, and let P be a p-subgroup of G containing subgroups $P_1, \ldots, P_n$ such that $P_i \subset G_i$. Let k be a large field of characteristic p and suppose that for each i there is a smooth geometrically connected G_i-Galois cover of $\mathbf{P}^1_k$ branched only at ∞, where P_i is an inertia group. Then there is a smooth geometrically connected G-Galois cover of $\mathbf{P}^1_k$ branched only at ∞, where P is an inertia group.*

Proof. Let $\tilde{G} \subset G$ be the subgroup generated by $G_1, \ldots, G_{n-1}$, and let $\tilde{P} \subset P$ be the subgroup generated by $P_1, \ldots, P_{n-1}$. Thus $\tilde{P} \subset \tilde{G}$, and $\tilde{P} \subset P$ is a p-group. Furthermore, G is generated by its subgroups $\tilde{G}$ and G_n, and P contains the subgroups $\tilde{P}$ and P_n. So by induction we are reduced to proving the case $n = 2$.

So we have a group G generated by subgroups G_1 and G_2, together with a G_1-Galois cover and a G_2-Galois cover of the line over k, each branched only over one point, and with P_1 and P_2 respectively occurring as inertia groups. We write these covers as $X \to U$ and $Y \to V$, where U and V are copies of $\mathbf{P}^1_k$ with affine parameters u and v. We may assume that the ramification occurs over the points $u = 0$ and $v = 0$, respectively, and we let $\xi \in X$, $\eta \in Y$ be points with inertia groups P_1, P_2, respectively. Thus $\operatorname{Spec} \hat{\mathcal{O}}_{X,\xi} \to \hat{U} := \operatorname{Spec} k[[u]]$ is a P_1-Galois cover, and $\operatorname{Spec} \hat{\mathcal{O}}_{Y,\eta} \to \hat{V} := \operatorname{Spec} k[[v]]$ is a P_2-Galois cover, each totally ramified over the closed point. Let $\hat{X} = \operatorname{Ind}_{P_1}^{P} \operatorname{Spec} \hat{\mathcal{O}}_{X,\xi}$ and $\hat{Y} = \operatorname{Ind}_{P_2}^{P} \operatorname{Spec} \hat{\mathcal{O}}_{Y,\eta}$; these are (disconnected) P-Galois covers of $\hat{U}$ and $\hat{V}$, respectively, whose generic fibers correspond to P-Galois ring extensions $k((u)) \subset \Omega_1$ and $k((v)) \subset \Omega_2$.

Let $R = k[[t,u,v]]/(uv - t^2)$ and $R' = R[1/(u + v - 2t)]$. Thus R is the complete local ring of a cone at its vertex $u = v = t = 0$, and R' corresponds to the open set obtained by deleting the line $u = v = t$. By the lemma, there is a connected normal P-Galois étale cover $C' \to D' = \operatorname{Spec} R'$ whose fiber over $t = 0$ has the property that its pullback to $k((u))$ agrees with that of $\hat{X}$ and its pullback to $k((v))$ agrees with that of $\hat{Y}$, and such that the normalization C of $D = \operatorname{Spec} R$ in C' is totally ramified over $(u = v = t)$.

Let Z be the result of identifying the point $(u = 0)$ in U with the point $(v = 0)$ in V, so that the two copies of the projective line meet transversally there. Thus Z is the closure in $(\mathbf{P}^1_k)^3$ of the locus of $uv = 0$ in $\mathbf{A}^3_k$, where the two lines meet at the origin ζ and we have an isomorphism F: $R/(t) \xrightarrow{\sim} \hat{\mathscr{O}}_{Z,\zeta}$. Applying Theorem 1(b) to $(Z,\{\zeta\})$ in the G-Galois case with $m = 0$ (i.e., no deformation variables), we conclude that there is a G-Galois cover ψ: $X \to Z$ with the property that its fibers over $\operatorname{Spec} \hat{\mathscr{O}}_{Z,\zeta}, U', V'$ agree with $\operatorname{Ind}_P^G C$, $\operatorname{Ind}_{P_1}^G X$, $\operatorname{Ind}_{P_2}^G Y$, respectively. That is, we have a G-Galois thickening problem for covers (cf. Section 3), where $X \to Z$ is unramified away from ξ, and where the generic fiber of $R_\zeta \subset A_\zeta$ is ramified just at a single $k((t))$-point, viz. $(u = v = t)$, over which P is an inertia group.

So by Theorem 5, there is a smooth projective k-curve $Z^\#$ of genus 0, and a smooth connected G-Galois branched cover of curves $X^\# \to Z^\#$ whose branch locus consists of exactly one k-point, over which P is an inertia group. Thus we may identify $Z^\#$ with $\mathbf{P}^1_k$ and ζ with the point at infinity; and the cover $X^\# \to \mathbf{P}^1_k$ is then as desired. ∎

Next we turn to the case of tamely ramified covers. Suppose that $\bar{Z}$ is a connected semistable curve over k. Let $\bar{Z}_1, \dots, \bar{Z}_N$ be the irreducible components of $\bar{Z}$, and let Z [resp. Z_i] be the normalization of $\bar{Z}$ [resp. of $\bar{Z}_i$]. Thus $\bar{Z}$ is the union of the $\bar{Z}_i$'s and Z is the disjoint union of the Z_i's. Let $\bar{S}$ be the singular locus of $\bar{Z}$, and let S be the inverse image of $\bar{S}$ under the normalization morphism ν: $Z \to \bar{Z}$. Thus ν restricts to a two-to-one morphism $S \to \bar{S}$. Let $\bar{S}_{ij} = \bar{S}_{ji}$ be the subset of $\bar{S}$ consisting of points whose two inverse images under ν respectively lie on Z_i and Z_j, and let $S_{ij} = \nu^{-1}(\bar{S}_{ij}) \cap Z_i$ and $S_i = \bigcup_{j=1}^N S_{ij}$. For every $\zeta \in S$, let $\sigma(\zeta)$ be the unique other element of S that has the same image under ν. Thus σ is a fixed-point-free involution of S that restricts to a bijection $S_{ij} \to S_{ji}$ for all i, j, and $\bar{Z}$ is the quotient of Z with respect to the identification of ζ with $\sigma(\zeta)$ for all $\zeta \in S$.

In the above situation, consider a G-Galois admissible cover $\overline{X} \to \overline{Z}$ (cf. Section 3). So in particular, $\overline{X}$ is a semistable curve. Let X be the pullback of $\overline{X}$ under $Z \to \overline{Z}$, and let $\tilde{X}_i \to Z_i$ be the pullback of $X \to Z$ under the inclusion $Z_i \hookrightarrow Z$. Thus X is smooth, and each $\tilde{X}_i \to Z_i$ is a G-Galois cover of the form $\mathrm{Ind}_{G_i}^G X_i \to Z_i$, where X_i is a connected component of $\tilde{X}_i$ and $G_i \subset G$ is the decomposition group of this component. Since the cover is admissible, it follows that the canonical generator of inertia at each point of $\tilde{X}_i$ over $\zeta \in S_{ij}$ is conjugate to the inverse of the canonical generator of inertia at each point of $\tilde{X}_j$ over $\sigma(\zeta)$.

Conversely, given $\overline{Z}$ as above, suppose that we are given a finite group G, subgroups $G_1, \dots, G_N \subset G$, and smooth connected G_i-Galois covers $X_i \to Z_i$ (whose branch loci are of the form $\Delta_i \cup S_i$, where Δ_i and S_i are disjoint). Let $\tilde{X}_i = \mathrm{Ind}_{G_i}^G X_i \to Z_i$ be the induced G-Galois cover, whose identity connected component may be identified with X_i. For each $\zeta \in S_{ij}$, pick a point $\xi_\zeta \in X_i$ lying over ζ, and let $g_\zeta \in G_i$ be the canonical generator of inertia of the G_i-Galois cover $X_i \to Z_i$ at ξ_ζ.

Writing $Z = \bigcup Z_i$ as above, and setting $X = \bigcup \tilde{X}_i$, we would like to descend the (disconnected) G-Galois cover $X \to Z$ to a connected admissible G-Galois cover $\overline{X} \to \overline{Z}$. As above, $\overline{Z}$ is obtained from Z by identifying each $\zeta \in S_{ij}$ with $\sigma(\zeta) \in S_{ji}$. Given elements $c_\zeta \in G$ such that $c_\zeta^{-1} = c_{\sigma(\zeta)}$ for all $\zeta \in S$, we may construct a curve $\overline{X}$ from X by identifying each $g(\xi_\zeta) \in \tilde{X}_i$ with $gc_\zeta(\xi_{\sigma(\zeta)}) \in \tilde{X}_j$ transversally, for all g, i, j, and all $\zeta \in S_{ij}$. Below we give a criterion for this $\overline{X}$ to give us the desired connected admissible G-Galois cover of $\overline{Z}$.

For each sequence $\mathbf{a} = a_0, \dots, a_r$ in $\{1, \dots, N\}$, let C_{a} be the set of elements $g \in G$ that can be written in the form $h_0 c_{\zeta_0} h_1 c_{\zeta_1} \cdots c_{\zeta_{r-1}} h_r$, where each $h_i \in G_{a_i}$ and each $\zeta_i \in S_{a_i, a_{i+1}}$. Let $C_G = C_G(\{G_i\}, \{c_\zeta\})$ be the union of all of the sets C_{a}, as $\mathbf{a}$ varies over sequences such that $a_0 = a_r = 1$. It is easily seen that C_G is a subgroup of G.

PROPOSITION 2. *In the above construction*:

(a) *The curve $\overline{X}$ is semistable if and only if $c_\zeta g_{\sigma(\zeta)} c_\zeta^{-1} \in \langle g_\zeta \rangle$ for every $\zeta \in S$. In this case, the cover $X \to Z$ descends to a G-Galois covering morphism $\overline{X} \to \overline{Z}$.*

(b) *The cover $X \to Z$ descends to an admissible G-Galois covering morphism $\overline{X} \to \overline{Z}$ if and only if $g_\zeta^{-1} = c_\zeta g_{\sigma(\zeta)} c_\zeta^{-1}$ for every $\zeta \in S$.*

(c) *If the above G-Galois cover $\overline{X} \to \overline{Z}$ is admissible, then $\overline{X}$ is connected if and only if $C_G = G$.*

Proof. (a) The curve $\overline{X}$ is semistable if and only if each point in $\tilde{X}_i$ over $\zeta \in S_{ij}$ is identified with a unique point of $\tilde{X}_j$ over $\sigma(\zeta) \in S_{ji}$ (where these points are necessarily distinct, even when $i = j$, since σ has no fixed points). Now as above, $g(\xi_\zeta) \in \tilde{X}_i$ is identified with $gc_\zeta(\xi_{\sigma(\zeta)}) \in \tilde{X}_j$, and vice versa. Interchanging the roles of i, j does not yield any additional identifications of a given point over $\zeta \in S_{ij}$, since σ is an involution and since $c_\zeta^{-1} = c_{\sigma(\zeta)}$. Now given a point $\xi \in \tilde{X}_i$ over $\zeta \in S_{ij}$, the element $g \in G$ such that $\xi = g(\xi_\zeta)$ is determined up to right multiplication by a power of g_ζ. But for any integer e, we have $gc_\zeta(\xi_{\sigma(\zeta)}) = gg_\zeta^e c_\zeta(\xi_{\sigma(\zeta)})$ if and only if $c_\zeta^{-1} g_\zeta^e c_\zeta$ stabilizes $\xi_{\sigma(\zeta)}$, i.e., if and only if $c_\zeta^{-1} g_\zeta^e c_\zeta$ is a power of $g_{\sigma(\zeta)}$. So we see that semistability is equivalent to this latter condition holding. By interchanging ζ and $\sigma(\zeta)$, we deduce that semistability is equivalent to $c_\zeta g_{\sigma(\zeta)} c_\zeta^{-1} \in \langle g_\zeta \rangle$.

If $c_\zeta g_{\sigma(\zeta)} c_\zeta^{-1} \in \langle g_\zeta \rangle$, so that $\overline{X}$ is semistable, then the morphism $X \to Z$ descends to a morphism $\overline{X} \to \overline{Z}$, since points being identified have the same image. Moreover, in this case $\overline{X} \to \overline{Z}$ is a G-Galois cover, since it is generically separable, and since the action of G on the orbit of ξ_ζ is compatible with the action on the orbit of $\xi_{\sigma(\zeta)}$.

(b) The cover $\overline{X} \to \overline{Z}$ in (a) is admissible if and only if it is semistable and the canonical generators of inertia at identified points are inverses. Now the point $g(\xi_\zeta)$ has canonical generator $gg_\zeta g^{-1}$, and $gc_\zeta(\xi_{\sigma(\zeta)})$ has canonical generator $(gc_\zeta)g_{\sigma(\zeta)}(gc_\zeta)^{-1}$. These elements are inverses if and only if $g_\zeta^{-1} = c_\zeta g_{\sigma(\zeta)} c_\zeta^{-1}$; and the condition in (a) is subsumed by that of (b). So the assertion in (b) follows.

(c) Let $\overline{X}_i$ be the image of X_i in $\overline{X}$, so that $\mathrm{Ind}_{G_i}^G \overline{X}_i$ is the image of $\tilde{X}_i = \mathrm{Ind}_{G_i}^G X_i$ in $\overline{X}$. Since $\overline{X} \to \overline{Z}$ is a cover and since $\overline{Z}$ is connected, it follows that every irreducible component of $\overline{X}$ is in the same connected component of $\overline{X}$ as some irreducible component of $\mathrm{Ind}_{G_i}^G \overline{X}_1$. So connectivity is equivalent to the condition that the irreducible components of $\mathrm{Ind}_{G_i}^G \overline{X}_1$ all lie in the same connected component of $\overline{X}$. Now two irreducible components Y, Y' of $\overline{X}$ lie in the same connected component of $\overline{X}$ if and only if there is a chain of irreducible components $Y_0, \ldots, Y_r$ of $\overline{X}$ respectively lying over irreducible components $\overline{Z}_{a_0}, \ldots, \overline{Z}_{a_r}$ in $\overline{Z}$, where $Y_0 = Y$ and $Y_r = Y'$, and where Y_{i-1} and Y_i meet over some point of $\overline{S}_{a_{i-1}, a_i}$, for each i. It now suffices to prove the following:

Claim. For $g, g' \in G$, the irreducible components $g(\overline{X}_i)$ and $g'(\overline{X}_j)$ can be connected by a chain of components of length r if and only if $g^{-1}g' \in \overline{G}_{\mathrm{a}}$ for some sequence $\mathbf{a} = a_0, \ldots, a_r$ with $a_0 = i$ and $a_r = j$.

Namely, if this claim is shown, then we may take $i = j = 1$, and g, g' arbitrary, and conclude that any two irreducible components of $\mathrm{Ind}_{G_i}^G \overline{X}_1$ lie in the same connected component of $\overline{X}$ if and only if $C_G = G$.

So it remains to prove the claim, and by induction we are reduced to the case $r = 1$. For this, we will verify that $g(\overline{X}_i)$ and $g'(\overline{X}_j)$ meet at a point over $\zeta \in \overline{S}_{ij}$ if and only if $g^{-1}g' = hc_\zeta h'$ for some $h \in G_i$ and $h' \in G_j$. Now $g(\overline{X}_i)$ and $g'(\overline{X}_j)$ meet over ζ if and only if there is a $g^* \in G$ such that $g^*(\xi_\zeta) \in g(\overline{X}_i)$ and $g^* c_\zeta(\xi_{\sigma(\zeta)}) \in g(\overline{X}_j)$. This is equivalent to the condition that $g^*(\overline{X}_i) = g(\overline{X}_i)$ and $g^* c_\zeta(\overline{X}_j) = g'(\overline{X}_j)$ for some $g^* \in G$, i.e., that $h := g^{-1}g^* \in G_i$ and $h' := (g^* c_\zeta)^{-1} g' \in G_j$. This in turn is equivalent to the desired condition that $hc_\zeta h' = g^{-1}g'$ for some $h \in G_i$ and $h' \in G_j$. ∎

Remark. In the framework of [Sa1] and [Sa2], a graph is constructed in which edges are associated to double points and vertices are associated to components. In that set-up, the group C_G above can be interpreted in terms of paths in the associated graph.

The above proposition shows that one obtains a connected G-Galois admissible cover of curves $\overline{X} \to \overline{Z}$ if we are given the following data:

- Smooth connected projective k-curves $Z_1, \dots, Z_N$ of genus $\gamma_1, \dots, \gamma_N$, and smooth connected G_i-Galois covers $X_i \to Z_i$, where each G_i is a subgroup of G.
- For each i, disjoint finite subsets $\Delta_i, S_i \subset Z_i$ of cardinalities d_i, s_i, such that $X_i \to Z_i$ is étale away from $\Delta_i \cup S_i$ and tamely ramified over S_i; and a canonical generator of inertia $g_\zeta \in G_i$ at a point over each $\zeta \in S_i$.
- A bijection $\sigma: S \to S$ with no fixed points, where $S = \bigcup S_i$; and for each i a partition $S_i = S_{i1} \cup \cdots \cup S_{iN}$ such that $\sigma(S_{ij}) = S_{ji}$ for all i, j.
- For each $\zeta \in S$, an element $c_\zeta \in G$ such that $g_\zeta^{-1} = c_\zeta g_{\sigma(\zeta)} c_\zeta^{-1}$, for which the associated group $C_G(\{G_i\}, \{c_\zeta\})$ (as in Proposition 2) is equal to G.

Combining the above proposition with the results of Section 3, we obtain:

THEOREM 7. *Let k be a large field, let G be a finite group, and suppose we are given data as above. Then there is a smooth projective k-curve $Z^\#$ of genus $\gamma = \sum_i(\gamma_i + s_i/2) - N + 1$ and a connected G-Galois cover $X^\# \to Z^\#$ that is étale except at $\sum_i d_i$ k-points, where respective inertia groups (and, in the tame case, canonical generators of inertia) are those of $X_i \to Z_i$ over the sets S_i.*

Proof. Let $\overline{Z}$ be the semistable curve obtained by taking the union of the curves Z_i after identifying each $\zeta \in S_{ij}$ with $\sigma(\zeta) \in S_{ji}$. Let $s_{ij} = \#S_{ij}$, so that $s_{ij} = s_{ji}$. Then the image $\overline{Z}_i$ of Z_i in $\overline{Z}$ is an irreducible semistable curve with $s_{ii}/2$ nodes, and whose normalization has genus γ_i; so $\overline{Z}_i$ has arithmetic genus $\gamma_i + s_{ii}/2$. So by [St1, Lemma 2.2], the arithmetic genus of $\overline{Z}$ is equal to $\sum_i(\gamma_i + s_{ii}/2) + \sum_{i<j} s_{ij} - N + 1 = \gamma$.

Let $\overline{X} \to \overline{Z}$ be the cover obtained as in the above construction, by patching the induced covers $\mathrm{Ind}_{G_i}^{G} \overline{X}_i \to \overline{Z}_i$ according to the given data. Then Proposition 2 asserts that $\overline{X} \to \overline{Z}$ is a connected G-Galois admissible cover of semistable curves. So the conclusion follows by Corollary 1 to Theorem 5. ∎

By [Gr2, XIII, Corollaire 2.12], if $G \in \pi_A(g)$ (i.e., if G is the Galois group of an étale cover of a curve of genus g), then there are generators $a_1, b_1, \ldots, a_g, b_g$ such that $\prod_{i=1}^{g}[a_i, b_i] = 1$. Moreover, the p-rank of G (i.e., the rank of the maximal p-quotient of G) is at most g. This raises the question of the extent to which the converse of these assertions will hold (and cf. [St2]). In particular, we have the following result, as a consequence of Theorem 7, by taking each $\Delta_i = \varnothing$ above:

COROLLARY 1. *Let k be a large field of characteristic p. Let G be a finite group generated by elements $a_1, b_1, a_2, b_2, \ldots, a_g, b_g$, where p is prime to the order of a_i for all i. Let*

$$G_0 = \langle a_1, b_1 a_1^{-1} b_1^{-1}, a_2, b_2 a_2^{-1} b_2^{-1}, \ldots, a_g, b_g a_g^{-1} b_g^{-1} \rangle,$$

and suppose that $G_0 \in \pi_A^t(\mathbf{P}_k^1 - \{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\})$, corresponding to a cover with description

$$\left(a_1, b_1 a_1^{-1} b_1^{-1}, a_2, b_2 a_2^{-1} b_2^{-1}, \ldots, a_g, b_g a_g^{-1} b_g^{-1}\right). \tag{$*$}$$

Then G lies in $\pi_A(g)$.

Proof. We will construct data as in the situation before the above theorem, and then apply that result to obtain the corollary.

Let $Z_0, \ldots, Z_g$ be copies of $\mathbf{P}_k^1$, let $S_0 = \{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\} \subset Z_0$, and for $i > 0$ let $S_i = \{0_i, \infty_i\}$ (the points at 0 and ∞ on this copy of the line). Let $X_0 \to Z_0$ be the tamely ramified G_0-Galois cover of X_0 branched at S_0, and with description $(*)$, guaranteed by the hypotheses of the corollary. For $i > 0$ let $X_i \to Z_i$ be a cyclic cover branched over S_i and having Galois group $G_i = \langle a_i \rangle$ and description (a_i^{-1}, a_i). On S_i for $i > 0$, let $g_{0_i} = a_i^{-1}$ and $g_{\infty_i} = a_i$. On S_0, let $g_{\alpha_j} = a_j$ and $g_{\beta_j} = b_j a_j^{-1} b_j$. For each $i > 0$ let $\Delta_i = \varnothing$. Let σ interchange $0_i \in Z_i$ with $\alpha_i \in Z_0$, and interchange $\infty_i \in Z_i$ with $\beta_i \in Z_0$. Let $S_{0i} = \{\alpha_i, \beta_i\}$ for $i > 0$; let $S_{i0} = \{0_i, \infty_i\}$ for $i > 0$; and otherwise let $S_{ij} = \varnothing$. For $i > 0$, let $c_{0_i} = 1$ and $c_{\infty_i} = b_i$, while on S_0 let $c_{\alpha_i} = 1$ and $c_{\beta_i} = b_i^{-1}$. The group $C_G(\{G_i\}, \{c_\zeta\})$ contains the elements $a_i \in G_i$ and the elements $b_i = c_{\infty_i}$; and hence $C_G = G$. So the conclusion follows from Theorem 7. ∎

In the next corollary, we give a family of groups for which all of the hypotheses of the preceding result are satisfied. When $g = 2$, this result already appears in [St2, Corollary 4.2], and the machinery developed above enables an easy proof in general.

COROLLARY 2. *Let k be a large field of characteristic p. Let G be a finite group generated by elements $a_1, b_1, a_2, b_2, \dots, a_g, b_g$, such that p is prime to the order of a_i for all i, $H = \langle a_1, a_2, \dots, a_g \rangle$ is normal in G, and* Out H *is trivial. Then G lies in $\pi_A(g)$.*

Proof. By normality, $b_i a_i^{-1} b_i^{-1} \in H$ for all i. Thus the group

$$G_0 = \langle a_1, b_1 a_1^{-1} b_1^{-1}, a_2, b_2 a_2^{-1} b_2^{-1}, \dots, a_g, b_g a_g^{-1} b_g^{-1} \rangle \subset G$$

is equal to H. So by Corollary 1, it suffices to show that there is a smooth connected H-Galois cover of $\mathbf{P}_k^1$ with description $(*)$ as in that corollary.

For every i, conjugation by b_i induces an automorphism of the normal subgroup H, and by hypothesis this automorphism is inner. So $b_i a_i^{-1} b_i^{-1}$ is a conjugate of a_i^{-1} by some element of H. Thus the above description is equivalent to the simpler description $(a_1, a_1^{-1}, a_2, a_2^{-1}, \dots, a_g, a_g^{-1})$, and it suffices to construct a cover of the projective k-line with this description.

This can be done as in [Ha1]. Namely, p is prime to n_i, where n_i is the order of a_i. So [Ha1, Proposition 3.4] implies that there exists a connected (locally standard) H-mock cover of $\mathbf{P}_k^1$ branched at g points $z = \alpha_1, \dots, \alpha_g$ with description $(a_1, \dots, a_g)$. Applying Theorem 5 above to the thickening problem given by extensions $R_i = k[[z - \alpha_i, t]] \subset A_i$, where A_i is the normalization of $R_i[y]/(y^2 - z(z - \alpha_i)^{n_i - 1})$, we obtain an H-Galois cover of the k-line with description $(a_1, a_1^{-1}, a_2, a_2^{-1}, \dots, a_g, a_g^{-1})$. ∎

Remark. (a) The above proof actually shows a bit more. Namely, rather than assuming that Out H is trivial, it suffices to assume that every inner automorphism of G restricts to an inner automorphism of H. This is equivalent to assuming that $G = H \cdot Z_G(H)$, where the second factor denotes the centralizer of H in G. In fact, even less is needed, viz. that $G = H \cdot Z_G(x)$ for all $x \in H$, since this also guarantees that $b_i a_i^{-1} b_i^{-1}$ is a conjugate of a_i^{-1} by some element of H.

(b) Alternatively, one may replace the hypotheses that H is normal in G and Out H is trivial by the single hypothesis that each commutator $[a_i, b_i]$ is trivial. For then we immediately have that $b_i a_i^{-1} b_i^{-1}$ is conjugate to a_i^{-1} in H, since they are equal.

COROLLARY 3. *Let k be a large field of characteristic p. Let G be a finite group generated by elements $a_1, b_1, a_2, b_2, \dots, a_g, b_g$, such that $\prod_1^g [a_i, b_i] = 1$. Assume that the subgroup $H = \langle a_1, a_2, \dots, a_g \rangle$ is normal in G and has order prime to p. Then G lies in $\pi_A(g)$.*

Proof. Since H is normal, each $b_i a_i b_i^{-1}$ lies in H. Since the order of H is prime to p, and since $\prod_1^g [a_i, b_i] = 1$, it follows from [Gr2, XIII, Corollaire 2.12] that there is a connected H-Galois cover of $\mathbf{P}_k^1 - \{\alpha_1, \beta_1, \dots, \alpha_g, \beta_g\}$ with description

$$\left(a_1, b_1 a_1^{-1} b_1^{-1}, a_2, b_2 a_2^{-1} b_2^{-1}, \dots, a_g, b_g a_g^{-1} b_g^{-1}\right).$$

Here p is prime to the order of each a_i, since $a_i \in H$. The conclusion now follows from Corollary 1. ▮

Remark. For the proof of Corollary 3, one does not need the full strength of the above hypotheses. Namely, one may replace the assumption that H is normal in G and has order prime to p by the weaker assumption that the group G_0 of Corollary 1 has order prime to p. For example, this holds if the normal closure of H has order prime to p.

As an opposite application of Theorem 7, we can take nonempty sets Δ_i and a construction in genus 0, to obtain Galois groups over open subsets of the line. For example, we have the following:

COROLLARY 4. *Let k be a large field of characteristic p. Let G be a finite group generated by elements a, b, c, d such that $abcd = 1$, and such that p does not divide the orders of the subgroups $\langle a, b\rangle$ and $\langle c, d\rangle$. Then for appropriate $\lambda \in k - \{0, 1\}$, the group G lies in $\pi_A^t(\mathbf{P}_k^1 - \{0, 1, \infty, \lambda\})$, corresponding to a cover of description (a, b, c, d).*

Proof. As in the proof of Corollary 1, we will construct data as in the situation just before the statement of Theorem 7, and then use that theorem to obtain the desired assertion.

Let $N = 2$, and let Z_1, Z_2 be copies of $\mathbf{P}_k^1$, of genus $\gamma_i = 0$. Let $G_1 = \langle a, b\rangle$ and let $G_2 = \langle c, d\rangle$. Let $\Delta_i = \{0_i, \infty_i\} \subset Z_i$ and let $S_i = \{1_i\} \subset Z_i$, where ζ_i denotes the point ζ on the line Z_i. Thus $d_i = 2$ and $s_i = 1$. Since the orders of G_1 and G_2 are prime to p, there are G_i-Galois covers $X_i \to Z_i$ with branch loci $\{0_i, \infty_i, 1_i\}$, and with respective descriptions $(a, b, b^{-1}a^{-1})$ and $(c, d, d^{-1}c^{-1})$. Thus $g_1 := b^{-1}a^{-1} = cd$ is a canonical generator of inertia over 1_1, and $g_2 := d^{-1}c^{-1} = ab = g_1^{-1}$ is a canonical generator of inertia over 1_2. Let $S_{ii} = \varnothing$ and let $S_{ij} = S_i$ for $i \neq j$; let $S = S_1 \cup S_2$; and define $\sigma\colon S \to S$ as the map that switches 1_1 and 1_2. Finally, let $c_\zeta = 1$ for $\zeta \in S$. So $C_G = C_G(\{G_i\}, \{c_\zeta\})$ consists of elements expressible in the form $h_0 h_1 \cdots h_r$, where each h_i is in either G_1 or G_2, with $h_0, h_r \in G_1$. Here the elements $a, b \in G_1$ are expressible by sequences of length 0, while $c, d \in G_2$ are expressible by sequences of length 1 (by taking $h_0 = 1$). So the group $C_G = G$, and the hypotheses before Theorem 7 are satisfied. Thus there is a smooth connected projective k-curve $Z^\#$ of genus $\sum_i(\gamma_i + s_i/2) - N + 1 = 0$ and a connected

G-Galois cover $X^{\#} \to Z^{\#}$ that is étale except at four k-points, with description (a, b, c, d). Since $Z^{\#}$ is of genus 0 and has k-points, it is isomorphic to $\mathbf{P}_k^1$. Applying a projective linear transformation, we may send the four branch points respectively to $0, 1, \infty, \lambda$ for some $\lambda \in k - \{0, 1\}$. ∎

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